NOTE ON A NOTE OF H. F. TUAN

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The following theorem is proved.

THEOREM. *If Z is a nilpotent matrix with elements in a field K, then the replicas of Z are those and only those matrices which are of the form $f(Z)$, where $f(x)$ is an additive*[1] *polynomial in* $K[x]$.

The concept of a replica was introduced by Chevalley,[2] who proved this theorem when K is of characteristic zero. The theorem was proved in general by H. F. Tuan[3] by elementary methods. The object of this note is to give a simplification of Tuan's proof; in particular, computations involving the specific form of Z are avoided.

If $h(x)$ is additive, then according as K is of characteristic 0 or p, $h(x)$ will have one of the two forms

$$tx, \qquad \sum_{j=1}^{m} t_j x^{p^j} \qquad (t, t_j \in K). \tag{1}$$

For if $h(x)$ had any other terms, then $h(x)+h(y)=h(x+y)$ would contain product terms $x^\alpha y^\beta$, $\alpha>0$, $\beta>0$. Conversely, polynomials of the form (1) are clearly additive. If $h(x)=\sum_{k=0}^{s} c_k x^k (c_k \in K)$, then we define

$$h^{[i]}(x) = \sum_{k=i}^{s} C_{k,i} c_k x^{k-i},$$

where the $C_{k,i}$ are binomial coefficients. Evidently

$$h^{(i)}(x) = i!\,h^{[i]}(x), \qquad h(x+y) = \sum_{i=0}^{s} h^{[i]}(x) y^i.$$

It follows from this that $h(x)$ is additive if and only if $c_0=0$ and $h^{[i]}(x)=c_i$ for $i>0$.

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[1] A polynomial $f(x)$ is additive if $f(x+y)=f(x)+f(y)$. The statement of the theorem in terms of the additivity of $f(x)$ rather than in terms of the explicit form (1), as well as the use of the derived polynomials $f^{[i]}(x)$ to replace explicit computation with binomial coefficients, was suggested by Professor Jacobson.

[2] Claude Chevalley, *A new kind of relationship between matrices*, Amer. J. Math. vol. 65 (1943) pp. 521–531. We make use of the definitions and notations of this paper.

[3] Hsio-Fu Tuan, *A note on the replicas of nilpotent matrices*, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 305–312, in particular Theorems (A) and (D).

If $f(x)$ is additive then it is easily seen that $f(Z)_{r,s}=f(Z_{r,s})$; hence (Z) is a replica of Z if $f(x)$ is additive. The converse follows from:[4]

LEMMA 1. *Let Z be a nilpotent matrix over a field K and let Z' be a matrix such that $Z'=f(Z)$, $Z'_{0,2}=g(Z_{0,2})$, where $f(x)$ and $g(x)$ are polynomials without constant terms. Then f may be assumed to be additive.*

This lemma does indeed imply the theorem since Chevalley[2] has proved (p. 529) that a replica Z' of Z satisfies the hypothesis of the lemma.

We now recall some definitions. If A and B are $n\times n$ matrices over K, then $A\times B$ is the $n^2\times n^2$ matrix formed by the $n\times n$ array of matrices $a_{ij}B$, where $A=(a_{ij})$.The following statements are evident:

$$(A \times B)(C \times D) = AC \times BD,$$
$$(A + A_1) \times B = A \times B + A_1 \times B,$$
$$cA \times B = c(A \times B) \quad \text{if} \quad c \in K,$$
$$A \times B = 0 \quad \text{implies} \quad A = 0 \quad \text{or} \quad B = 0. \tag{2}$$

Finally, $Z_{0,2}$ is defined as $Z\times E+E\times Z$; E is the $n\times n$ unit matrix, where n is the dimension of Z.

Since Z is nilpotent and since it may be assumed that $Z\neq 0$ (for Lemma 1 is trivial if $Z=0$), there is an integer m such that

$$Z^m \neq 0, \qquad Z^{m+1} = 0, \qquad 1 \leqq m \leqq n-1. \tag{3}$$

LEMMA 2. *If $A_0, A_1, \cdots, A_m$ are $n\times n$ matrices such that*

$$A_0 \times E + A_1 \times Z + \cdots + A_m \times Z^m = 0,$$

then $A_0=A_1=\cdots=A_m=0$.

PROOF. Multiplying by $E\times Z^m$, we obtain $A_0\times Z^m=0$, $A_0=0$ by (2) and (3). Multiplying successively by $E\times Z^{m-1}$, $E\times Z^{m-2}$, $\cdots$, we obtain $A_1=A_2=\cdots=0$.

PROOF OF LEMMA 1. In view of (3) the polynomial $f(x)$ may be assumed to be of degree at most m. We show that it must then be additive. Now

$$(Z_{0,2})^k = (Z \times E + E \times Z)^k = \sum_{i=0}^{k} C_{k,i} Z^{k-i} \times Z^i.$$

It follows that $(Z_{0,2})^{2m+1}=0$, so that $g(x)$ may be assumed of degree at most $2m$. We observe incidentally that Lemma 2 implies that

[4] Loc. cit., Theorems (B) and (C).

$(Z_{0,2})^m \neq 0$; if K is of characteristic 0, then also $(Z_{0,2})^{2m} \neq 0$, for $(Z_{0,2})^{2m} = C_{2m,m} Z^m \times Z^m$.

Now we have

$$
\begin{aligned}
Z'_{0,2} &= g(Z_{0,2}) = g(Z \times E + E \times Z) = \sum_{i=0}^{2m} g^{[i]}(Z \times E)(E \times Z)^i \\
&= g(Z) \times E + g^{[1]}(Z) \times Z + g^{[2]}(Z) \times Z^2 + \cdots \\
&\quad + g^{[m]}(Z) \times Z^m.
\end{aligned} \tag{4}
$$

On the other hand, placing $f(x) = \sum_{i=1}^{m} a_i x^i$, we have

$$
\begin{aligned}
Z'_{0,2} &= Z' \times E + E \times Z' = f(Z) \times E + E \times f(Z) \\
&= f(Z) \times E + a_1 E \times Z + a_2 E \times Z^2 + \cdots + a_m E \times Z^m.
\end{aligned} \tag{5}
$$

A comparison of (4) and (5) gives, by Lemma 2,

$$
\begin{array}{lll}
g(Z) = f(Z), & g^{[i]}(Z) = a_i E, & i = 1, \cdots, m; \\
g(x) \equiv f(x)(x^{m+1}), & g^{[i]}(x) \equiv a_i(x^{m+1}), & i = 1, \cdots, m.
\end{array}
$$

From the first congruence, $g^{[i]}(x) \equiv f^{[i]}(x)(x^{m+1-i})$, and from the second, $f^{[i]}(x) \equiv a_i(x^{m+1-i})$. Since $f^{[i]}(x)$ is of degree at most $m-i$, $f^{[i]}(x) = a_i$. By a previous remark it follows that $f(x)$ is additive, and this completes the proof.

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