40. A. R. Schweitzer: *On the genesis of number systems.* II.

In continuation of the preceding paper, other developments of number systems are: (2) in terms of reflexive, symmetric and transitive relations $\alpha\beta R\gamma\delta(\alpha+\beta=\gamma+\delta)$ and $\kappa\lambda P\mu\nu(\kappa\times\lambda=\mu\times\nu)$ in analogy with the author's relation $\alpha\beta K\gamma\delta$ (ibid. p. 394) and (3) as an elaboration of linear order based on the author's system ${}^1R_1$ (ibid. p. 378). In system (2) the author employs the binary relational symbol $\alpha\beta RP\gamma\delta$ to express $R\alpha\beta=P\gamma\delta$ or $\alpha+\beta=\gamma\times\delta$; $\alpha\beta RR\gamma\delta$ and $\kappa\lambda PP\mu\nu$ are assumed respectively equivalent to $\alpha\beta R\gamma\delta$ and $\kappa\lambda P\mu\nu$. Further assumptions are: 1. $\alpha\beta RP\gamma\delta$ implies $\gamma\delta PR\alpha\beta$. 2. $\alpha\beta XY\lambda\mu$, $\lambda\mu YZ\gamma\delta$ imply $\alpha\beta XZ\gamma\delta$, where X, Y, Z are on the set (R, P). 3. There exist uniquely in S the distinct elements ω (zero) and ϵ (unity) such that $\alpha\omega RP\alpha\epsilon$ for any α in S. 4. $\alpha\omega RP\beta\epsilon$ implies $\alpha=\beta$. 5. For any α, β in S there exist uniquely γ, δ such that $\alpha\beta R\gamma\omega$ and $\alpha\beta P\delta\epsilon$. 6. $\alpha\beta R\xi\omega$ and $\beta\gamma R\eta\omega$ imply $\alpha\eta R\xi\gamma$; and similarly for the relation P. Correspondingly, distributive, inversive and commutative properties are stated. The extension of the preceding system S in analogy with the author's systems ${}^nR_n$ and ${}^nK_n$ $(n=1, 2, 3, \cdots)$ is discussed. (Received October 19, 1945.)

Statistics and Probability

41. Will Feller: *Note on the law of large numbers and "fair" games.*

An example is exhibited to show that a game can be "fair" in the sense that the expectation of loss vanishes, and nevertheless such that the probability tends to one that after n trials there will be a positive loss L_n; the ratio of L_n to the accumulated entrance fees tends to zero as slowly as one pleases. On the other hand, in the classical Petersburg game entrance fees can be determined so that the game becomes fair in the sense that the probable loss or gain will be of smaller order of magnitude than the accumulated entrance fees. (To appear in the Annals of Mathematical Statistics.) (Received October 4, 1945.)

42. Will Feller: *On the normal approximation to the binomial.*

The goodness of the normal approximation to $T_{\lambda,\nu}=\sum_{k=\lambda}^{\nu} C_{n,k}p^k(1-p)^{n-k}$ is studied with particular reference to the practically important cases of small $np(1-p)$ and of comparatively large λ and ν. Limits of the integral are determined which depend quadratically on λ and ν and are such that the integral will approximate $T_{\lambda,\nu}$ from above or from below. The relative error is also studied. In a sense this paper continues a well known series of studies by Serge Bernstein (the latest in Izvestia Akademiia Nauk SSR, 1943). By the departure from the classical, but arbitrary, use of moments unexpected simplifications are obtained which render S. Bernstein's results more accurate and valid under less stringent conditions. (To appear in the Annals of Mathematical Statistics.) (Received October 4, 1945.)

43. P. R. Halmos: *The theory of unbiased estimation.*

Let $F(P)$ be a real-valued function defined on a subset E of the set D of all probability distributions on the real line. A function f of n real variables is an unbiased estimate of F if for every system $X_1, \cdots, X_n$ of independent random variables with the common distribution P the expectation of $f(X_1, \cdots, X_n)$ exists and equals $F(P)$, for all P in E. Under the assumption that E contains all purely discontinuous distributions, the class of all functions $F(P)$ which possess an unbiased estimate is characterized and all unbiased estimates of each such F are described. It is shown that there is in every case a unique symmetric unbiased estimate which coincides also with the unbiased estimate of least variance. Thus the classical estimates of the mean and the variance are justified from a new point of view, and also computable estimates of all higher moments are presented. It is interesting to note that for n greater than 3 neither the sample nth moment about the sample mean nor any constant multiple thereof is an unbiased estimate of the nth moment about the mean. (Received October 6, 1945.)

44. Isaac Opatowski: *Markoff chains and Tchebychev polynomials.*

Let the possible states be $0, 1, \cdots, n+1$ and the only transitions possible during any time dt $(i-1 \rightarrow i)$ for $1 \leqq i \leqq n+1$ and $(i+1 \rightarrow i)$ for $0 \leqq i \leqq n-1$. Let the conditional probabilities for these transitions be respectively $k_i dt + o(dt)$ and $g_i(dt) + o(dt)$, where $k_i = k$ for $1 \leqq i \leqq n$, $k_{n+1} = k$ or 0, $g_i = g$ or 0, k and g being two positives constants. The probability $P(t)$ of the existence of the state n at a time t if the state 0 existed at $t=0$ is in the general case a convolution of particular functions $P(t)$ corresponding to the following cases: (i) $k_{n+1}=0$, $g_i = g$ $(i \leqq n-1)$; (ii) $k_{n+1}=0$, $g_i = g$ $(i \leqq n-2)$, $g_{n-1}=0$; (iii) $k_{n+1}=k$, $g_i=g$ $(i \leqq n-1)$. In (i), $p(s) = \int_0^\infty e^{-st} P(t) dt = (k/g)^{n/2} / [s U_n(x)]$, where $U_n(x)$ is the Tchebychev polynomial of second kind and x is a linear function of s. The roots of U_n give an explicit expression of $P(t)$ as a linear combination of n exponentials whose coefficients are calculated in a form convenient for computations. In cases (ii) and (iii), $[p(s)]^{-1}$ is a linear combination of two U_i's and the roots of $[p(s)]^{-1}$ are located within narrow ranges, which makes the calculation of $P(t)$ possible within any accuracy desired. These chain processes occur in some biophysical phenomena and the paper will appear in Proc. Nat. Acad. Sci. U.S.A. under a slightly different title. (Received October 11, 1945.)

TOPOLOGY

45. Lipman Bers and Abe Gelbart: *A remark on the Lebesgue-Sperner covering theorem.*

A new and elementary proof is given of a somewhat stronger form of the well known Lebesgue-Sperner covering theorem (Math. Ann. vol. 70 p. 166; Abh. Math. Sem. Hamburgischen Univ. vol. 6 p. 265). Some corollaries are discussed. (Received October 19, 1945.)

46. R. H. Bing: *Solution of a problem of J. R. Kline.*

It is shown that a locally connected, compact, metric continuum S is topologically equivalent to the surface of a sphere provided no pair of points separates S but every simple closed curve separates S. On the assumption that an arc separates S, a simple closed curve is constructed that does not separate S. (Received October 10, 1945.)

47. O. G. Harrold: *The* ULC *property of certain open sets.* I. *Euclidean domains.*

Let M be a compact continuum which separates Euclidean 3-space. If M is deformation-free into a complementary domain A and $p^1(M)=0$, then the fundamental group of A vanishes. By means of this: if M^* is a compact continuum separating 3-space which is deformation-free into a complementary domain A, then A is ULC. If, in addition, $p^1(M^*)=0$ and A is bounded, this implies $\bar{A}$ is a singular 3-cell by a result of S. Eilenberg and R. L. Wilder (Amer. J. Math. vol. 64 (1942) pp. 613–622).