7. Ernst Snapper: *Polynomial matrices in one variable, differential equations and module theory.*

This paper establishes the foundation for the theory of matrices $A=(\alpha_{ij})$, where $(\alpha_{ij})\in P[x_1, \cdots, x_n]$. Part I treats the case $n=1$. Contrary to the classical procedure which uses sub-determinants of A, the theory is developed intrinsically in terms of the column space C and row space R of A. The meanings of the irreducible factors and multiplicities of the norm and elementary divisor of A for C and R thus become clear. Systems of linear differential equations and algebraic equations are fully discussed. Part II reviews and extends the ideal theoretic module theory, developed by P. M. Grundy in *A generalization of additive ideal theory*, Proc. Cambridge Philos. Soc. vol. 38 (1942), and by the author in *Structure of linear sets*, Trans. Amer. Math. Soc. vol. 52 (1942). This theory is the foundation for the case $n>1$. A general theory of systems of linear equations over any ring $\mathfrak{r}$ is developed. All known criteria for the solvability of such systems for special rings are corollaries of the *criterion of lengths* of this general theory. If $\mathfrak{r}=P[x]$, the theory becomes the theory of Part I. (Received October 7, 1945.)

8. Ernst Snapper: *Polynomial matrices in several variables.*

This paper discusses the theory of matrices $A=(\alpha_{ij})$, where $\alpha_{ij}\in P[x_1, \cdots, x_n]$. The module theory, discussed in Part II of the author's paper *Polynomial matrices in one variable, differential equations and module theory*, associates several invariants to the column space C and the row space R of A, for example the associated primes $\mathfrak{p}_j$, the $\mathfrak{p}_j$-lengths, the $\mathfrak{p}_j$-elementary divisors, and so on. Since R and C are *polynomial* modules, the theory of the Hilbert characteristic function can be developed for them which gives rise to one further invariant, called the $\mathfrak{p}_j$-degree. In terms of these invariants, the theory of the system of linear partial differential equations and algebraic equations, represented by A, is investigated. Furthermore, the irreducible factors and multiplicities of the norm and elementary divisor of A, as defined by the author in *The resultant of a linear set*, Amer. J. Math. vol. 66 (1944), are explained in terms of the above invariants. (Received October 7, 1945.)

Analysis

9. N. R. Amundson: *On the boundary value problem of third kind for the quasi-linear parabolic differential equation.*

The author considers the quasi-linear parabolic equation with boundary conditions of the *third* kind for the open rectangle, that is, $u_{xx}=f(x, y, u, p, q)$; $-a_1u_x+b_1u=c_1(y)$, when $x=0$; $a_2u_x+b_2u=c_2(y)$, when $x=l$; $u=\phi(x)$, when $y=0$, where $c_i(y)$ and $\phi^{(iv)}(x)$ are continuous and b_i/a_i are non-negative constants. By use of the Green's function for the problem the above system is shown to be equivalent to a nonlinear integro-differential equation. Assuming that $f(x, y, u, p, q)$ is continuous in all five variables, and that its partial derivatives with respect to y, u, p, q satisfy a Lipschitz condition in u, p, q and are bounded, the *existence* of a solution $u(x, y)$ of the integro-differential equation is proved by an iteration method. Under the further assumption the u_x and u_y satisfy a Hölder condition with respect to y, the *uniqueness* of the solution $u(x, y)$ is established. M. Gevrey (*Thèse*, Journal de mathématique (6) vol. 9 (1913) and vol. 10 (1914)) considers the same differential equation for boundary conditions of the first kind. (Received October 19, 1945.)

10. E. W. Barankin: *On the eigen-values of infinite matrices and of linear integral equations.* Preliminary report.

In a recent paper (Bull. Amer. Math. Soc. vol. 51 (1945)) the author established several symmetric upper bounds for the characteristic roots of a finite matrix. The methods there employed are carried over to the discretely infinite and continuous cases, to give, under weak restrictions to insure convergence, the corresponding results. For an infinite matrix the bounds are of the same form (with some small exceptions) as in the finite case. For the general homogeneous linear integral equation $\int_0^\infty K(s, t)f(t)dt = \lambda f(s)$ the bounds obtained are $|\lambda|^2 \leqq \text{l.u.b.}_{(s)} \{\int_0^\infty |K(s, t)| dt \cdot \int_0^\infty |K(t,s)| dt\}$, $|\lambda| \leqq \text{l.u.b.}_{(s)} \int_0^\infty |K(s, t)| dt$, $|\lambda| \leqq \text{l.u.b.}_{(s)} \int_0^\infty |K(t, s)| dt$, and certain generalizations of the first of these. (Received November 17, 1945.)

11. E. M. Beesley and A. P. Morse: *ϕ-Cantorian functions and their convex moduli.*

A set function ϕ^* may be associated with a function ϕ which is nondecreasing on $[0, \infty)$ with $\phi(0+) = \phi(0) = 0$. A set $A \subset I$ is said to be ϕ-Cantorian if it is the intersection of a denumerable sequence of sets each of which is associated with a family of intervals satisfying certain conditions involving ϕ and I. If $f(x) = \phi^*((-\infty, x] \cdot A)$ where A is a ϕ-Cantorian set, then f is called a ϕ-Cantorian function. The convex modulus Φ of a function f has the property that $|f(t_2) - f(t_1)| \leqq \Phi(t_2 - t_1)$ and satisfies certain other conditions. Methods for construction of ϕ-Cantorian sets and for determination of the convex modulus are considered. These methods are employed to construct in any interval a set of Lebesgue measure zero which cannot be covered by any sequence of intervals whose lengths are equal to a prescribed sequence of numbers whose sum is less than the length of the original interval. A symmetric product measure is constructed and a perfect plane set is exhibited which has measure one but which cannot be expressed as a sum of a denumerable sequence of sets of finite measure after being subjected to a properly chosen shear or rotation. (Received November 19, 1945.)

12. Lipman Bers and Abe Gelbart: *A topological property of solutions of partial differential equations.*

Given a function $u(x, y)$ defined in a domain D and satisfying the elliptic differential equation $(au_x + bu_y)_x + (bu_x + cu_y)_y = 0$, where a, b, c are analytic functions of x and y; then there exists a homeomorphism of D into a domain Δ of the ξ, η-plane which takes u into a harmonic function of ξ and η. The proof is based on Stöilow's topological characterization of analytic functions of a complex variable. (Received October 19, 1945.)

13. D. G. Bourgin: *A class of generating functions.*

This note continues previous work on orthonormal sequences of the type $\{f(nx)\}$. Among other things it is shown that if the associated $\phi(z)$ belongs to K' and has a finite base, then $\phi(z)$ is a quasi elementary solution. (Received October 19, 1945.)

14. D. G. Bourgin: *Complete sets of functions.*

(A) Let $\{g_n(x)\}$ be O.N. and complete in $L_2(E)$. Let $F_n(x) = f_n(x) - g_n(x)$. Then, under certain minor convergence conditions, a sufficient condition for completeness of $\{f_n(x)\}$ is that the Grammian of $\{F_n(x)\}$ have a bound inferior to 1. In particular, the restriction $\sup_{1\leq i<\infty}\sum_j |[F_i, F_j]| < 1$ is sufficient. (B) If $g_n(x) = \sin nx$ and $f_n(x) = f(nx)$ with $f(x) \sim \sin x - \sum b_i \sin ix$, then a sufficient condition for completeness of $\{f_n(x)\}$ in the space of odd functions in $L_2(-\pi, \pi)$ is essentially that $\sum [F_i, F_j] \cos (\log i - \log j)x \leq \theta < 1$ where the sum is taken over all relatively prime integers i and j. If only a finite number of b_i's are nonvanishing, θ can be 1. Some theorems of Szász type are also given. (Received October 18, 1945.)

15. R. H. Cameron and W. T. Martin: *The orthogonal development of nonlinear functionals in series of Fourier-Hermite functionals.*

The authors show that certain products of Hermite functions of orthogonal linear functionals form a closed orthonormal set in the space L_2^* of nonlinear (that is, not necessarily linear) functionals of the Wiener integrable square. In terms of this set, each functional F of L_2^* can be developed in an orthogonal series which converges to F in the mean of L_2^*. (Received October 19, 1945.)

16. Herman Chernoff: *Complex solutions of partial differential equations.* I.

The author considers certain classes $\mathcal{C}$ of complex solutions of equations $\Delta u + Au_x + Bu_y + Cu = 0$. Bergman demonstrated certain properties of these classes (see Bergman, Trans. Amer. Math. Soc. vol. 53 (1943) pp. 130–155 and vol. 57 (1945) pp. 299–331). The author investigates the distribution of b-points of functions $u \in \mathcal{C}$. He introduces in the usual manner the notion of the index of a b-point (that is, generalized concept of multiplicity) and denotes by $n[r, (u-b)^{-1}]$ the sum of the indices of points where $u = b$ in $|z| < r$ $(z = x+iy)$. Let $2\pi m[r, u] = \int_0^{2\pi} \log^+ |u| \, d\theta$ where $z = re^{i\theta}$. Using Nevanlinna's Second Fundamental Theorem, the author gives an upper bound for $m[r, u]$ in terms of $\sum_{\nu=1}^{q} n[r, (u-\alpha_\nu)^{-1}]$. This upper bound holds for a certain restricted set of positive r. However in certain cases it is shown that r belongs to this set if r is large enough and $u \neq \alpha_\nu$ on $|z| = r$. For certain subsets of functions $u \in \mathcal{C}$, it is proved (in analogy of the Picard Theorem) that u attains all finite values. (Received November 14, 1945.)

17. A. E. Heins and Norbert Wiener: *A generalization of the Wiener-Hopf integral equation.*

A method is given which enables one to find the solutions of the integral equations of the type (*) $f(x) = \lambda \int_0^\infty K(x+y) f(y) dy$ where $K(x) = O(e^{-x})$, $x \to \infty$, and $O(1/x)$, $x \to 0$. This is accomplished by decomposing (*) into an infinite sequence of bilateral faltung integral equations, each of which depends on the solution of the previous one. The final result appears as an infinite series of integral operators, applied to a known function. An example for which the final answer appears in closed form is given. (Received November 13, 1945.)

18. Mark Kac: *Distribution of eigenvalues of certain integral equations with an application to roots of Bessel functions.* Preliminary report.

Let $\rho(u) \in L(-\infty, \infty)$, $\rho(u) = \rho(-u)$ and let $F(\xi) = \int_{-\infty}^{\infty} \rho(u) \cos u\xi du$ be also in $L(-\infty, \infty)$. Consider the eigenvalues $\lambda_1(a), \lambda_2(a), \cdots$, of the integral equation (*) $\int_{-a}^{a} \rho(s-t) f(t) dt = \lambda f(s)$. It is shown that for $n \geq 2$ the limit of $(2a)^{-1} \sum \lambda_j^n$, as $a \to \infty$, is $\pi^{-1} \int_0^\infty F^n(\xi) d\xi$. Denoting by $N(\alpha, \beta; a)$ the number of eigenvalues of (*) which fall within (α, β) (not containing 0) it is easy to calculate explicitly $\lim_{a\to\infty}(2a)^{-1}N(\alpha, \beta; a)$ in terms of $F(\xi)$. If $\gamma > 0$ and $\lambda_1(\gamma)$, $\lambda_2(\gamma)$, $\cdots$, are the eigenvalues of the integral equation (**) $\int_{-\infty}^{\infty} \exp(-\gamma|s|)\rho(s-t)\exp(-\gamma|t|)f(t)dt = \lambda f(s)$, then the limit of $\gamma\sum\lambda_j^n$, as $\gamma\to 0$, is $(\pi n)^{-1}\int_0^\infty F^n(\xi)d\xi$. Denoting by $N(\alpha, \beta; \gamma)$ the number of eigenvalues of (**) which fall within (α, β) it is again easy to calculate $\lim_{\gamma\to 0}\gamma N(\alpha, \beta; \gamma)$ in terms of $F(\xi)$. If in (**) one puts $\rho(u)' = 2^{-1}\exp(-|u|)$ the eigenvalues are expressible in terms of roots of $J_{1/\gamma}(x)$ and $J_1'{}_{/\gamma}(x)$. As a consequence one obtains the following result: If $N(\alpha; \nu)$ denotes the number of positive roots of $J_\nu(x)$ which are less than $\alpha\nu$ $(\alpha > 1)$ then $\nu^{-1}N(\alpha; \nu) \to \pi^{-1}\{(\alpha^2-1)^{1/2} - \text{arc tan } (\alpha^2-1)^{1/2}\}$. This in turn implies, among others, that if $r_\nu(\nu)$ is the νth positive root of $J_\nu(x)$, then $\nu^{-1}r_\nu(\nu) \to x_1^2+1$, where x_1 is the first positive root of $\tan x = x$. (Received October 18, 1945.)

19. Charles Loewner: *On pairs of quadratic forms in Hilbert space.*

One may ask which of the infinitely many linear transformations in n-space transforming two given positive definite forms $A(x, x) = \sum_1^n a_{\rho\sigma}x^\rho x^\sigma$, $B(x, x) = \sum_1^n b_{\rho\sigma}x^\rho x^\sigma$ into each other is the most "economical." The answer will depend on the method by which the economy is measured. It is natural to introduce a metric in the space of linear transformations and to call that transformation the most economical which is closest to the identity. Again, it is natural to use a metric that is intrinsically connected with the two given forms. A positive definite form represents a metric in the vector space. In a well known manner one can derive from it a metric in an arbitrary tensor space, especially in the space of linear transformations. It is remarkable that the two minimum problems derived in this way from the given forms have the same unique solution. Its matrix T_0 can be expressed by the matrices A and B of the given forms as that square root $(A^{-1}B)^{1/2}$ whose characteristic values are all positive. These elementary considerations can be generalized to Hilbert space by a suitable modification of the extremum problem. It can be shown that a suitable $(A^{-1}B)^{1/2}$ has similar extremum properties as in the finite-dimensional case. (Received October 22, 1945.)

20. L. B. Robinson: *Solution of an integral equation by alternating successive approximations.* I.

In the integral equation $u(x) = \lambda\sum_{i=1}^3 (P_{in}(x)/Q_{in}(x))\int_0^x(1+s^2)^{-1}(\overline{P}_{in}(s)/\overline{Q}_{in}(s)) \cdot u(s^{-p})ds + P_{4n}(x)/Q_{4n}(x) + \sum_{1=1}^3 u_{0i}(P_{in}(x)/Q_{in}(x))$, P and Q are polynomials of order n and p is an integer. The adjoint is $u(x^{-1}) = \lambda\sum_{i=1}^3(P_{in}(x^{-1})/Q_{in}(x^{-1}))\int_0^x -(1+s^2)^{-1} \overline{P}_{in}(s^{-1})/Q_{in}(s^{-1})u(s^{+p})ds + P_{4n}(x^{-1})/Q_{4n}(x^{-1}) + \sum_{=1} V_{0i}(P_{in}(x^{-1})/Q_{in}(x^{-1}))$. By alternating successive approximations obtain $u(x)$ and $u(x^{-1})$ and then weld them together by calculating the v_0 as linear functions of the u_0. The u_0 may be all zero. The author made his calculations using the number 3 but 3 can be replaced by any integer. (Received November 2, 1945.)

21. L. B. Robinson: *Solution of an integral equation by alternating successive approximations.* II.

Write $u(x) = \lambda\int_0^x(1+s^2)^{-1}G(x, \pi s/(1+s))u(s^{-2})ds + F(x)$, $\pi s/(1+s) = \sigma$, $\pi/(1+s) = \tau$. G, F are finite in the real domain with period π, $-\pi$. Then $u(x) = \lambda\int_0^x(1+s^2)^{-1}\{a_0(x) + \sum_{m=1}^\infty[a_m(x)\cos m\sigma + b_m(x)\sin m\sigma]\}u(s^{-2})ds + F(x)$, $u(x^{-1}) = \lambda\int_0^x -(1+s^2)^{-1}\{a_0(x^{-1}) + \sum_{m=1}^\infty[a_m(x^{-1}) \cos m\tau + b_m(x^{-1}) \sin m\tau]\}u(s^{+2})ds + F(x^{-1}) + \sum_{m=1}^\infty[a_m(x^{-1})u_m + b_m(x^{-1})v_m]$. Solve by alternating successive approximations. $u(x)$ and $u(x^{-1})$ converge within the unit circle. The two can be welded together if the constants u_m, v_m

are selected properly. The results can be extended to the complex domain. (Received November 2, 1945.)

22. Robert Schatten (National Research Fellow) and John von Neumann: *The cross-space of linear transformations.* II.

The present notation is that of Bull. Amer. Math. Soc. Abstract 51-9-162. For any crossnorm $\alpha \geqq \lambda$, $(\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2)^*$ may be considered as a Banach space of linear transformations from $\mathfrak{B}_1$ into $\mathfrak{B}_2^*$ (from $\mathfrak{B}_2$ into $\mathfrak{B}_1^*$), while $\mathfrak{B}_1^* \otimes_{\alpha'} \mathfrak{B}_2^* \subset (\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2)^*$ may be considered as the Banach space of all those linear transformations of $(\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2)^*$ which may be approximated in norm by linear transformations with finite-dimensional range. In particular for a Hilbert space $\mathfrak{H}$, $\mathfrak{H} \otimes_\gamma \mathfrak{H} = (\mathfrak{H} \otimes_\lambda \mathfrak{H})^*$ represents the "trace-class," that is, all linear transformations T on $\mathfrak{H}$ with finite trace; the norm $\gamma(T) =$ trace $(T'T)^{1/2}$, where T' represents the adjoint of T. It is also shown that λ does not necessarily represent the least crossnorm, that is, there exist crossnorms whose associates are not crossnorms. (Received November 13, 1945.)

23. Menahem Schiffer: *Hadamard's formula and variation of domain-functions.*

Hadamard's formula for the variation of the Green's function $g(x, y)$ for a varying domain D holds for smooth boundaries only. Considering special variations of D and using Green's identity, one may express this variation in terms of g and its derivatives at interior points of D only, resulting in a formula valid for general domains D. This was previously derived by the author in another way (Amer. J. Math. vol. 65 (1943) pp. 341–360). The same variation formula holds for many other domain-functions, for example, log $|f(x)|$, where $f(x)$ maps D on the exterior of a circle slit along concentric arcs and radial "stretches." In the case of smooth boundaries, the formula may be transformed into one of Hadamard's type, but is different for each of the domain-functions mentioned. The general formula is of wide applicability in extremum problems of conformal representation. Utilizing the periods of the analytic function with real part $g(x, y)$, one obtains variation formulae for the harmonic measures and the capacity constants of the different boundary continua. The method also permits applications in the theory of Riemann surfaces. The variation of the elementary integrals and their periods satisfy very similar formulae for a general class of deformations. (Received October 22, 1945.)

24. C. F. Stephens: *Solutions of systems of nonlinear difference equations in the neighborhood of a singular point.*

Consider the system of nonlinear difference equations (1) $y_i(x+1) = x^k \sum_{j=1}^n b_{ij}(x) y_j(x) + x^k f_i(y_1(x), \cdots, y_n(x); x)$ where the f_i begin with terms of the second degree in $y_i(x)$, are analytic functions of $y(x)$, and continuous functions of x in the neighborhood of $(x = \infty, y_1(x) = 0, \cdots, y_n(x) = 0)$. k is taken to be a positive integer and $f_i(0, \cdots, 0; x) \equiv 0$. By making use of the transformation $y_i(x) = [\Gamma(x)] z_i(x)$, where $\Gamma(x)$ is the well known gamma function, and the earlier results of the author (Bull. Amer. Math. Soc. Abstract 50-5-148), one shows that there exist many solutions of equations (1). These solution functions are continuous functions of x in a certain domain extending to infinity on the left and approach zero as a limit as x approaches infinity on rays parallel to the negative axis of reals. (Received October 18, 1945.)

25. S. E. Warschawski: *On the modulus of continuity of the mapping function at the boundary in conformal mapping.*

The author proves the following theorem: Let D be a simply connected region such that (i) D contains the unit circle and is contained in $|w|<R$; (ii) if D is divided into two parts by a crosscut of diameter $\delta<1$, then the diameter d of the subregion of D which does not contain the origin satisfies the inequality $d\leqq\mu\delta+\eta$ (μ and η are constants, $\mu\geqq 1$, $\eta\geqq 0$). Suppose that $w=f(z)$ maps $|z|<1$ conformally onto D ($f(0)=0, f'(0)>0$). Let k be an arbitrary constant, $k\geqq 4R^2$, $k>\mu^2$. If z_0 is any point on $|z|=1$ and if z_1 and z_2 are in $|z|<1$ with $|z_i-z_0|\leqq r\leqq\exp[-k\pi^2/4]$, then $|f(z_1)-f(z_2)|\leqq(kr^\alpha/\mu)+\eta k^{1/2}/(k^{1/2}-\mu)$ where $\alpha=(2/(\pi^2k))(\log k-2\log\mu)$. This result is applied to the following problem. Let C_1 and C_2 be closed Jordan curves containing $w=0$ in their interiors D_i, such that C_2 is in the ϵ-neighborhood of C_1 (that is, every point of C_2 is within a circle of radius ϵ about some point of C_1) and C_2 in the ϵ-neighborhood of C_1. Let $w=f_i(z)$ map $|z|<1$ onto D_i ($f_i(0)=0, f_i'(0)>0$). Then a function $\Phi(\epsilon)$ is determined, which, aside from ϵ, depends only on certain parameters characterizing the C_i, such that $|f_1(z)-f_2(z)|\leqq\Phi(\epsilon)$ for $|z|\leqq 1$. (Received October 19, 1945.)

26. J. W. T. Youngs: *Various definitions of surface and area.* Preliminary report.

This paper lists several definitions of the word "surface" and uses recent results in the field to show how the term "area" can be applied to each. The principal result is that in each case the area is a lower semi-continuous function of the surface. (Received October 17, 1945.)

APPLIED MATHEMATICS

27. Edward Kasner and John DeCicco: *Heat surfaces.*

If a region of space is heated by conduction, the temperature ν at a time t at a point (x, y, z) is $\nu=\phi(x, y, z, t)$, where ϕ satisfies the Fourier heat equation. Kasner has introduced the term heat surfaces for those along which $\nu=$const. and $t=$const. In general, there are ∞^2 heat surfaces. In the present work, the authors extend to space certain theorems of Kasner concerning heat families in the plane, published in 1932–1933, Proc. Nat. Acad. Sci. U.S.A. There are no systems of ∞^2 planes or ∞^2 spheres which form a heat family except in the imaginary domain. The only sets of ∞^1 planes which form a heat family are the pencils. A system of ∞^1 spheres is a heat family if and only if it is a concentric set. The only isothermal systems of planes are pencils and the only isothermal sets of spheres are concentric families. The cases where there are only ∞^1 heat surfaces are connected with the equations of Laplace, Poisson, and Helmholtz-Pockels. Finally, these results are extended to n dimensions. (Received October 11, 1945.)

28. H. E. Salzer: *Coefficients for repeated integration with central differences.*

The present paper is aimed toward facilitating double or k-fold repeated quadrature of a function which is tabulated at a uniform interval, with its central differences of even order (see Abstract 51-9-172). When the Everett interpolation formula is integrated k times over an interval of tabulation, one obtains a formula for stepwise multiple quadrature in the form (1) $\int_{x_0}^{x_1}\cdots\int_{x_0}^{x}\int_{x_0}^{x}f(x)(dx)^k=h^k[A_0^{(k)}f_0+B_0^{(k)}f_1$