A NOTE ON THE FIRST NORMAL SPACE OF A V_m IN AN R_n

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Let N be the normal plane at a point p of a surface V_2 in a Euclidean 4-space R_4. Calapso[2] proved that the hypersphere S in R_4 passing through p and with center c in N cuts V_2 in a curve with a double point at p, at which the two tangents to the curve coincide if and only if c lies on the Kommerell conic. The Kommerell conic is the locus of the point in which N (at p) is cut by the neighboring normal planes of V_2.

The purpose of this note is to generalize this result to the case of a subspace V_m in a Euclidean n-space R_n. To do this we shall first state some definitions and known results concerning the first (or principal) normal space of V_m in R_n.[3]

Let X^k $(k=1, \cdots, n)$ be the rectangular cartesian coordinates in R_n and let

$$X^k = x^k(u^a) \qquad (a, b, c = 1, \cdots, m) \tag{1}$$

be the equations of a V_m. Put

$$\overset{k}{B}_a = \partial_a \overset{k}{x} \equiv \partial x^k / \partial u^a. \tag{2}$$

Then the fundamental tensor and curvature tensor of V_m in R_n are, respectively,

$$'g_{cb} = \sum_k \overset{k}{B}_c \overset{k}{B}_b, \tag{3}$$

$$H_{cb}^{\cdot\cdot k} = \partial_c \overset{k}{B}_b - {}'\Gamma^a_{cb} \overset{k}{B}_a, \tag{4}$$

where $'\Gamma^a_{cb}$ is the Christoffel symbol of the second kind for V_m.

Let us consider the figure surrounding a certain point p of V_m. We have at p a tangent m-plane and a normal $(n-m)$-plane N. Let i^a be the unit tangent vector at p of an arbitrary curve in V_m passing through p. Then the component in N of the first curvature vector of the curve with respect to R_n is

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[2] R. Calapso, *Sulle reti di Voss di uno spazio lineare quadri dimensionale*, Rendiconti Seminario matematico Roma (4) vol. 2 (1938) pp. 276–311.

[3] See J. A. Schouten and D. J. Struik, *Einführung in der neueren Methoden der Differentialgeometrie* II, Groningen, 1938, chap. 3; D. Perepelkine, *Sur la courbure et les espaces normaux d'une V_m dans R_n*, Rec. Math. (Mat. Sbornik) N.S. vol. 42 (1935) pp. 81–100.

(5) $$u^k = H_{cb}^{\cdot\cdot k}\, i^c i^b.$$

The vector u^k spans the *first* normal m'-plane N' (in N) of V_m in R_n.

The arithmetic mean of the vector u^k for m mutually orthogonal directions i^a is the *mean* normal curvature vector

(6) $$M^k = {}'g^{cb} H_{cb}^{\cdot\cdot k}/m.$$

Any vector n_k $(=n^k)$ in N orthogonal to N' is such that

(7) $$n_k H_{cb}^{\cdot\cdot k} = 0.$$

p is called a *semi-umbilical* point if there exists a vector v_k such that

(8) $$v_k H_{cb}^{\cdot\cdot k} = {}'g_{cb}$$

is satisfied. Because of (7) we may suppose that v_k is a vector in N'.

The normal $(n-m)$-plane at the neighboring point $p+dp$ may or may not intersect N' (at p) at points other than p depending on the direction of dp. But we call the locus of the intersection of N' at p (the point p being excluded) by the normal $(n-m)$-planes of all the neighboring points $p+dp$ the *K-variety* at p of V_m in R_n. The equation of the K-variety is

(9) $$\text{Det}\,(Y_k H_{cb}^{\cdot\cdot k} - {}'g_{cb}) = 0,$$

where Y_k is a variable vector in N'. The K-variety is an algebraic hypersurface of order m in N'. At a semi-umbilical point, it is a hypercone in N' with vertex at the end point $v(x^k+v^k)$ (cf. (8)).

We are now in a position to prove the following theorems.

THEOREM 1. *The hypersphere S in R_n passing through p and with center at a point c in N intersects V_m in a V_{m-1} with a singular point at p. The tangent lines to V_{m-1} at p form a hypercone C (in the tangent m-plane to V_m) of generally the second degree.*

THEOREM 2. *p is semi-umbilical if and only if there exists a hypercone C at p which is of at least the third degree.*

THEOREM 3. *All the points in N with the same projection in N' have the same hypercone C. There exist two points in N' having the same hypercone C if and only if p is semi-umbilical. At a semi-umbilical point all the points (with the exception of the point v (x^k+v^k)) on each line in N' passing through v have the same hypercone C. No two points in N' noncollinear with v have the same hypercone C.*

THEOREM 4. *The locus of the point in N' whose hypercone C sustains* *an orthogonal ennuple of generators is the polar hyperplane of the end point of the mean normal curvature vector with respect to the unit hypersphere in N' (with center at p).*

THEOREM 5. *The K-variety is the locus of the point in N' whose hypercone C has a line of vertices.*

Theorem 5 for $m=2$, $n=4$, $N'=N$ reduces to the theorem of Calapso quoted at the beginning of this paper.

PROOF. The expansion of $x^k(u^a)$ in the neighborhood of $p: x_0^k = x^k(u_0^a)$ is

$$x^k = x_0^k + (\partial_a x^k)_0 du^a + 2^{-1}(\partial_c\partial_b x^k)_0 du^c du^b + \cdots .$$

But by (2) and (4),

$$\partial_a x^k = B_a^k, \qquad \partial_c\partial_b x^k = \partial_c B_b^k = H_{cb}^{\cdot\cdot k} + {}'\Gamma_{cb}^a B_a^k.$$

Therefore

$$x^k = x_0^k + (B_a^k)_0 du^a + 2^{-1}(H_{cb}^{\cdot\cdot k} + {}'\Gamma_{cb}^a B_a^k)_0 du^c du^b + \cdots . \tag{10}$$

The equation of the hypersphere S in R_n passing through p and with center at a point $c(x_0^k+c^k)$ in N is

$$\sum_k (X^k - x_0^k - c^k)^2 = \sum_k (c^k)^2.$$

Using (1) for X^k we see that S intersects V_m at the points $(u_0^a+du^a)$ at which

$$\sum_k [- c^k + (B_a^k)_0 du^a + 2^{-1}(H_{cb}^{\cdot\cdot k} + {}'\Gamma_{cb}^a B_a^k)_0 du^c du^b + \cdots]^2 = \sum_k (c^k)^2. \tag{11}$$

Let us arrange this in powers of du^a. Then the constant term disappears. The first term vanishes because c^k is orthogonal to the tangent m-plane spanned by $(B_a^k)_0$:

$$\sum_k c^k (B_a^k)_0 = 0. \tag{12}$$

The second degree term is

$$\left[\sum_k (B_c^k B_b^k)_0 - \sum_k c^k (H_{cb}^{\cdot\cdot k} + {}'\Gamma_{cb}^a B_a^k)_0\right] du^c du^b$$
$$= ({}'g_{cb} - c_k H_{cb}^{\cdot\cdot k})_0 du^c du^b$$

by (3) and (12). This proves Theorem 1 and gives us the equation of the hypercone C as

$$(c_k H_{cb}^{\cdot\cdot k} - 'g_{cb}) Z^c Z^b = 0, \tag{13}$$

where Z^a is a variable direction in the tangent m-plane at p.

Theorem 2 follows at once from (8), (13) and (11).

The hypercones at p for the two distinct points c $(x_0^k + c^k)$ and d $(x_0^k + d^k)$ are the same if and only if a constant ρ exists such that

$$(d_k - \rho c_k) H_{cb}^{\cdot\cdot k} = (1 - \rho)\, 'g_{cb} \tag{14}$$

is satisfied. If d and c have the same projection in N' we have by (7)

$$(d_k - c_k) H_{cb}^{\cdot\cdot k} = 0. \tag{15}$$

Therefore (14) will be satisfied by $\rho = 1$. Hence all the points in N with the same projection in N' have the same hypercone C.

This shows that we may confine our attention to the points in N' for the consideration of the hypercone C. Let this be done. Then (15) can no longer hold, and condition (14) cannot be satisfied by $\rho = 1$. Consequently, (14) may be put into the form (8) with

$$v_k = (d_k - \rho c_k)/(1 - \rho). \tag{16}$$

Therefore if there exist two points in N' with the same hypercone C then p must be a semi-umbilical point. Conversely, at a semi-umbilical point the locus of the point in N' whose hypercone C is the same as that of the point c (distinct from v) is the straight line cv minus the point v.

The hypercone (13) sustains an orthogonal ennuple, that is, contains m mutually orthogonal generators if and only if

$$'g^{cb} (c_k H_{cb}^{\cdot\cdot k} - 'g_{cb}) = 0,$$

that is, by (6) if

$$c_k M^k = 1. \tag{17}$$

Theorem 4 is an immediate consequence of this.

The hypercone (13) has a line of vertices if and only if

$$\operatorname{Det}\, (c_k H_{cb}^{\cdot\cdot k} - 'g_{cb}) = 0, \tag{18}$$

that is, if c lies on the K-variety (9). This proves Theorem 5.

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