THE HYPERSURFACE CROSS RATIO

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Introduction. In my note on *A* 5 *curve theorem generalizing the theorem of Carnot*,[1] I introduced the notion of curve cross ratio. This extension of the ordinary cross ratio is the simplest situationally invariant (3.4) case of the generalized or hypersurface cross ratio of $n+1$ pairs of hypersurfaces in n-space which is the subject of the following lines. The generalized cross ratio is at the same time a generalization of the resultant of $n+1$ quantics; the connection between cross ratios and resultants occurred to me when reading a paper of P. Humbert.[2]

The properties of the generalized cross ratio, including extensions of some of those of the ordinary cross ratio, will be developed, together with the similar and interdependent properties of an analogous generalization of the intersection of n hypersurfaces to pairs of hypersurfaces, in §3. This section, much of the contents of which is known, is parallel to §§1 and 2 on the ordinary resultant and intersection. In each section, after the definition and fundamental properties, the influence of a rational transformation of coordinates and of permutation, variation and linear combination of the hypersurfaces is studied.

1. **The resultant.** 1.1. DEFINITION. Let $x=(x_0, \cdots, x_n)$ be a point in (complex, or algebraically closed) projective n-space, $n \geqq 0$, and $a=(a_0, \cdots, a_n)$ a system of $n+1$ quantics of positive degrees $\bar{a}_0, \cdots, \bar{a}_n$ in the variables $x_0, \cdots, x_n$. Then the resultant $[a]$ is an irreducible polynomial in the coefficients of a with $[x_k^{\bar{a}_k}]=1$, such that $[a]=0$ if, and only if, an $x \neq 0$ with $a(x)=0$ (that is, $a_0(x)=0, \cdots, a_n(x)=0$ exists. The resultant is unique since the conditions of irreducibility and $[x_k^{\bar{a}_k}]=1$ distinguish it from its powers and multiples respectively.[3]

1.2. *Degree.* $[a]$ is a quantic of degree $\prod_k \bar{a}$ in the coefficients of a_k. Considering $\bar{a}_k$ as degree of the coefficients, we can write $[a]\cdot=\prod \bar{a}$.

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[1] Bull. Amer. Math. Soc. vol. 51 (1945) pp. 972–975.

[2] *Sur l'orientation des systèmes de droites*, Amer. J. Math. vol. 10 (1888) pp. 258–281.

[3] Cf. van der Waerden, *Moderne Algebra* vol. 2, 1931, pp. 18–21. As $|\ |$ is used for the absolute value, $[\ ]$ (already generally used in case of the vector product) is preferable for determinants, resultants and intersections. The product of all degrees shall be denoted by $\prod \bar{a}$, and if $\bar{a}_k$ is omitted, by $\prod_k \bar{a}$ ($=1$ for $n=0$, as all void products including 0 degree in 3.4 and 3.8).

1.3. *Multiplication.* If one of the given quantics a_k is a product $a_k=\prod a_k^{(i)}$, we have $[a]=\prod_i[a_0, \cdots, a_{k-1}, a_k^{(i)}, a_{k+1}, \cdots, a_n]$, in short $[\cdots, \prod, \cdots]=\prod[\ ]$, as both sides vanish simultaneously, are of equal degree, and can be simultaneously equal to 1.

1.4. *Special cases.* If $a_k\equiv 0$, then $[a]=0$.

For $\prod\bar{a}=1$, $[a]$ is a determinant.

For $n=1$, $[a]$ is a product of $\prod\bar{a}$ determinants.

Further note the *monomial* case $[\alpha_k x_k^{\bar{a}_k}]=\prod\alpha_k^{\Pi_k\bar{a}}$.

1.5. *Rational transformation.* Let $f_0, \cdots, f_n$ be quantics of equal degree $\bar{f}>0$, then $y=f(x)$ defines a rational transformation. If, for $x\neq 0$, also $y\neq 0$, that is, if $[f]\neq 0$, the transformation is called *regular*; in it, to every y, there belong $\bar{f}^n$ points x.[4] Hence, the only regular birational transformations are the projective transformations.

If $a(y)=0$ for $x\neq 0$, then either $y\neq 0$, $[a]=0$, or $y=0$, $[f]=0$. Hence $[a(f)]$ must be of the form $c[a]^r[f]^s$, $\bar{c}=0$ (that is, c is constant). In the monomial case $a_k=\alpha_k x_k^{\bar{a}_k}, f_k=\beta_k x_k^{\bar{f}}$ we have $a_k(f)=\alpha_k\beta_k^{\bar{a}_k}x_k^{\bar{f}\bar{a}_k}$, $[a(f)]=\prod(\alpha_k\beta_k^{\bar{a}_k})^{\bar{f}^n\Pi_k\bar{a}}$. We obtain $[a(f)]=[a]^{\bar{f}^n}[f]^{\Pi\bar{a}}$.

1.6. *Permutation.* If the quantics a_k are permuted, then $[a]$ and the new resultant $[a']$ are irreducible and of equal degree and vanish together, so $[a']=c[a]$, $\bar{c}=0$. Choosing quantics which are products of linear forms, we see by 1.3 and the laws of determinants that $c=(\pm 1)^{\Pi\bar{a}}$ for a permutation of sign ± 1.

1.7. *Variation.* If $\bar{a}_k\geqq\bar{a}_j$, $j\neq k$, and if a_k is replaced by $a_k+\lambda a_j$, where λ is a quantic of degree $\bar{a}_k-\bar{a}_j$, then $[a]$ and the new resultant $[a']$ are of equal degree and vanish together, so $[a']=c[a]$, $\bar{c}=0$. Choosing $\lambda=0$, we see that $c=1$.

1.8. *Linear combination.* If several quantics a_k of equal degree are replaced by linear combinations $\sum l_{jk}a_k$, $\bar{l}_{jk}=0$, then by repeated application of 1.7 and 1.2 the new resultant is $[a']=[l]^{\Pi_k\bar{a}}[a]$.

If all degrees $\bar{a}_k$ are equal, 1.6–1.8 are special cases of 1.5.

2. **The intersection.** 2.1. DEFINITION. If $a_1, \cdots, a_n$, $n\geqq 1$, are fixed quantics of positive degree, so that the intersection $a_1=\cdots=a_n=0$ of n hypersurfaces consists only of a finite number of points P_i (for $n=2$, quantics without a common factor of positive degree), then for $\bar{l}=1$, $[l, a]=0$ only if some $l(P_i)=0$; hence $[l, a]=\prod l(P_i)$ with appropriate multiplicities of sum $\prod\bar{a}$ and with a constant factor according to which the coordinates of the points P_i are determined. The system of these P_i, whose order may be changed and whose coordinates may be multiplied by constants, with product 1, will be de-

[4] By Bézout's theorem, if the number of points is finite. But for, say, $y_0\neq 0$, any variety of positive dimension of points x with $y_0f_k(x)-y_kf_0(x)=0$ would have a point in common with $f_0(x)=0$, against the assumption of regularity.

noted by $P=[a]$ (the same symbol as for the resultant, but referring to n quantics only); we consider the intersection P as product of its points and write accordingly $[l, a]=l([a])$.

For a general a_0, $[a_0, a]$ and $a_0([a])$ (that is, $\prod a_0(P_i)$) are of equal degree and vanish together, so they differ only by a constant factor c. Choosing a_0 as product of linear forms, we see by 1.3 that $c=1$, so $[a_0, a]=a_0([a])$ (well known for $n=1$).

2.2. *Degree.* If a_k is replaced by ca_k, $\bar{c}=0$, then $[a]$ becomes $c^{\Pi_k \bar{a}}[a]$; we may again say that $[a]$ is of degree $\prod_k \bar{a}$ in the coefficients of a, and, as in 1.2, write $[a]\cdot=\prod \bar{a}$, so the degree of an intersection is the number of its points. The degree of a resultant in one of the hypersurfaces is the number of meets of the others.

2.3. *Multiplication.* By $[l, a]=l([a])$ and 1.3 we see that again $[\cdots, \prod, \cdots]=\prod[\ ]$, the latter symbol denoting juxtaposition (union) of the points of the respective intersections.

2.4. *Special cases.* If $[a]$ is defined, then no a_k is equal to 0.

For $\prod \bar{a}=1$, $[a]$ consists of n determinants ("vector product").

In the monomial case we have $[\alpha_k x_k^{\bar{a}_k}]=\prod \alpha_k^{\Pi_k \bar{a}}(1, 0, \cdots, 0)$.

2.5. *Rational transformation.* Assume $[f]\neq 0$, and $[a(f)]$ definable by 2.1. A point Q_i belongs to $[a(f)]$ if $f(Q_i)$ belongs to $[a]$, so we may write $[a(f)]=[a]:f$. To obtain the exact degree and factor of P in $f(Q)$, put $l=x_k$, $l(f)=f_k$, then, by 2.1, the formula of 1.5 becomes $f_k(Q)=(x_k(P))^{f^n}[f]^{\Pi \bar{a}}=x_k(P^{f^n})[f]^{\Pi \bar{a}}$, or $f([a]:f)=P^{f^n}[f]^{\Pi \bar{a}}$; f^n agrees with 1.5, fourth line.

Rational variety. If $a_{m+1}=\cdots=a_n=0$ is a fixed rational variety $f(t)$, $t=(t_0, \cdots, t_m)$, of degree $\bar{f}$ such that to different t there belong different points $f(t)$, then the cross ratios $[a]$ and $[a(f)]$ as functions of the coefficients of $a_0, \cdots, a_m$ vanish together and are of equal degree, which is the number of meets of the hypersurfaces except $a_k=0$, so they differ only by a constant depending on f and on the form $c^{\Pi \bar{a}}$ (product up to $\bar{a}_m$). A factor can be given to f so that $c=1$.

For $n=1$, that is, a unicursal curve, $[a(f)]$ is a product of determinants.

If $a_{m+1}=\cdots=a_n=0$ is composed of several rational varieties, $[a]$ equals the product of the corresponding cross ratios. For $m=0$ this is again the final formula of 2.1.

2.6. *Permutation.* For a permutation of sign ± 1, 2.1 and 1.6 give $a'=(\pm P_i)$.

2.7. *Variation.* By 2.1 and 1.7, $[\cdots, a_k+\lambda a_j, \cdots]=[a]$.

2.8. *Linear combination.* By 2.7 and 2.2, $[a']=[l]^{\Pi_k \bar{a}}[a]$.

3. Generalized cross ratio and intersection. 3.1. DEFINITION. If

$a_0, \cdots, a_n$ are quantics of positive degree, except a_k, which is a quotient $b_k:c_k$ of quantics of positive degree, we may, in agreement with 1.3, define $[a]$ as $[a_0, \cdots, a_{k-1}, b_k, a_{k+1}, \cdots, a_n]$ $:[a_0, \cdots, a_{k-1}, c_k, a_{k+1}, \cdots, a_n]$, and $\lambda b_k:\lambda c_k$, $\bar{\lambda}>0$, will give the same value. Likewise, if every a_k is a homogeneous rational function $b_k:c_k$, $[a]$ is defined as quotient of products of 2^n resultants. This "resultant of rational functions" we call also *cross ratio* of the corresponding $n+1$ pairs of hypersurfaces $b_k=0$ and $c_k=0$.[5] It may be 0, ∞, or indeterminate. Artificial indetermination, due to the choice of λ, must be avoided.

The intersection may equally be defined for rational functions, in agreement with 2.3, by admitting negative multiplicities of points P_i, or, in case all multiplicities are 0, a number without point. And the notions of resultant and intersection remain linked by the relation $[a_0, a]=a_0([a])$. Here, for $\bar{a}_0=\bar{b}_0-\bar{c}_0=0$, every factor $a_0(P_i)$ equals $d_i:e_i$, if P_i is on the curve $e_ib_0-d_ic_0=0$ of the pencil (b_0, c_0), d_i and e_i being constants.

3.2. *Degree.* $[a]$ is of degree $\prod_k\bar{a}$ in the coefficients of b_k, and of degree $-\prod_k\bar{a}$ in the coefficients of c_k; so we can again write $[a]=\prod\bar{a}$. The same statements hold for an intersection $[a]$ of pairs of hypersurfaces.

3.3. *Multiplication.* Again, for cross ratios and intersections, $[\cdots, \prod, \cdots]=\prod[\ ]$; and for both, $[\cdots, 1:a_k, \cdots]=1:[a]$.

3.4. *Special cases.* No b_k or c_k is allowed to be congruent to 0.

In the monomial case, the relations of 1.4 and 2.4 subsist.

If $\bar{a}_k=0$, $[a]$ is of degree 0 in the coefficients of all other quantics; if $\bar{a}_j=\bar{a}_k=0$, $[a]$ depends only on the *situation* of the given hypersurfaces.[6] In this case, by 3.1, the cross ratio $[a]$ is a product of "cross ratios $a_k(P_i:Q_i)=a_k(P_i):a_k(Q_i)$ of a pair of points and a pair of hypersurfaces of equal degree," another situational invariant.[7] As $b_k(P_i:Q_i)$ is the ratio of the first and last coefficient of z in $b_k(z_0P_i+z_1Q_i)$, that is, the product of the ratios $-z_1:z_0$ corresponding to the meets of $b_k=0$ with the line P_iQ_i, we see that $a_k(P_i:Q_i)$ equals a product of ordinary cross ratios on P_iQ_i, each of which is determined by a meet of $b_k=0$ and a meet of $c_k=0$.

For cross ratios and intersections with constant a_k, $[a]=a_k^{\Pi_k\bar{a}}$, and with constant a_j and a_k, $[a]=1$. Thus, in contrast to 1.1, the

[5] Or of the *virtual* hypersurfaces $a_k=0$, van der Waerden, *Algebraische Geometrie*, 1939, p. 179.

[6] If $\bar{b}_k>0$, $\bar{c}_k>0$, we can attain $\bar{a}_k=0$ by replacing b_k and c_k by their own powers.

[7] Special cases of which are the ratio in Newton's theorem on algebraic curves, and, passing to the logarithms, the angle, and Laguerre's "orientation."

resultant of quantics of non-negative degree may be reducible; and it may be equal to 0 even if an $x \neq 0$ with $a(x)=0$ exists, which happens if there are 2 or more constant a_k, all of them equal to 0 (this case, it is true, has been excluded). Yet, in formal developments, we shall admit either or both $\bar{b}_k=0$ and $\bar{c}_k=0$, so we may assume b_k and c_k without a common factor of positive degree.

3.5. *Rational transformation.* The relations of 1.5 and 2.5 subsist. We see that, for $\prod \bar{a}=0$, the cross ratio and intersection are projective invariants. They are not invariant to dualization, common points occurring without common tangents and vice versa.

For a general (not necessarily regular) rational transformation and $\bar{a}_0=0$, $a_0(P_i)$ in 3.1 remains unaltered; thereby the change of the cross ratio and the intersection can be obtained.

Rational variety. The conclusions of 2.5 subsist; hence, for $\prod \bar{a}=0$, $[a]=[a(f)]$. For a unicursal curve $f(t)$ with inhomogeneous t, met by $b_0=0$, $c_0=0$, $b_1=0$, $c_1=0$ respectively in the systems of points $f(\beta_0), f(\gamma_0), f(\beta_1), f(\gamma_1)$ or, admitting negative multiplicities, by $a_0=0$ and $a_1=0$ (of degrees 0) in $f(\alpha_0)$ and $f(\alpha_1)$, we obtain $[a]=\alpha_0-\alpha_1 =(\beta_0-\beta_1)(\gamma_0-\gamma_1):(\beta_0-\gamma_1)(\gamma_0-\beta_1)$ where $\beta_0-\beta_1$, and so on, denote products of differences. For a line and $\bar{b}_0\bar{c}_0\bar{b}_1\bar{c}_1=1$, this is the ordinary cross ratio.

It would be interesting to know whether similar laws hold for non-unicursal curves.

3.6. *Permutation.* The relations of 1.6 and 2.6 subsist. Interchanging of b_k and c_k gives, by 3.3, the reciprocal value (or negative multiplicities). All other permutations may change more than the sign of $[a]$, or the sign of the multiplicities. Thus a cross ratio in $(n-1)$-space, or an intersection in n-space, $n>1$, has not $(2n)!$ but (generally) $N=(2n)!:n!2^n=1\cdot 3\cdots(2n-1)$ values, and with the reciprocals $2N$ values. Change of sign occurs only if among the $2n$ hypersurfaces there are n of even and n of odd degree; these must be combined in pairs, which can be done in $n!$ ways, so in this case, together with the opposites, there are respectively $N+n!$ or $2N+2n!$ values.

But the N values are not independent, the number of resultants (or intersections) being only $C_{2n,n}=(2n)!:n!^2$, that is, 6, 20, 70, 252, 924, 3432, 12870, 48620, $\cdots$ instead of 3, 15, 105, 945, 10395, 135135, 2027025, 34459425, $\cdots$. Moreover, a depends only on the $C_{2n,n}-1$ ratios of resultants, and for odd n only on the $C_{2n,n}:2$ ratios of a resultant of certain hypersurfaces and that of all other hypersurfaces.

The true sequence of numbers of independent values begins

2, 5, 14, $\cdots$ as seen below. The quantics will be denoted by 1, 2, $\cdots$, $2n$. Cross ratios and intersections are denoted as in the following example. If the given $[a]$ is $[2:7, 3:5, 6:1, 4:8]$, we bring 1 to the first place: $\pm 1:[a]=[1:6, 2:7, 3:5, 4:8]$. In the second place we see 6, which is the 5th number after 1; so we begin our symbol by 5. The remaining numbers 234578 are now ordered cyclically, beginning after 6, that is, with 7. 7 is brought to the next place, and we write $\pm a=[1:6, 7:2, 3:5, 4:8]$. The figure after 7, that is, 2, is the second in cyclic order; our symbol begins therefore 62. 3 is already in the proper place, 5 is second among 458; hence (622). 8 is not in its place, so $\pm 1:[a]=(622)$.

$n=2$. We have $(1)=[1:2,3:4]=[13][24]:[14][23]$, $(2)=[1:3,4:2]$, $(3)=[1:4,2:3]$. Multiplying, we get A: $(1)(2)(3)=(-1)$ to the power $(\bar{2})(\bar{3})+(\bar{2})(\bar{4})+(\bar{3})(\bar{4})$, hence there are 2 independent values.[8]

$n=3$. Permuting 345 cyclically, we have, as before, A : $(11)(12)(13)=\pm 1$. 234 gives B: $(11)(23)(31)=\pm 1$. 235 gives $(12)(23)(43)=\pm 1$, which follows also from the two preceding types A and B ("types" means that the first figures of the symbols may be replaced by their cyclic successors). So the 5 values (γ_1) determine the others; they are independent, as, for example, (11) contains [145] which occurs in no other (γ_1).

$n=4$. 234 gives C: $(111)(251)(311)=\pm 1$. 235 gives $(123)(221)(451)=\pm 1$, or, by B and C, D: $(221)=\pm(111)(131)(311)(511)$. 236 gives $(131)(213)(511)=\pm 1$, or, by B and C, again D. 237 gives $(143)(212)(651)=\pm 1$, or, by ABC, again D. Finally, 246 gives $(123)(353)(512)=\pm 1$, or, by ABC, a relation E containing (341). By C, D, E respectively, all $(\delta 51)$, $(\delta 21)$, $(\delta 41)$ are expressed by the 14 values $(\delta 11)$, $(\delta 31)$. These are independent, as, for example, (131) contains [1567] which occurs in no other $(\delta 31)$ or $(\delta 11)$, and (111) contains [1457] which occurs in no other $(\delta 11)$.

Every relation for $[a]$ and its permutations holds also if one of the quantics is 1, that is, for $2n-1$ hypersurfaces.[9] The converse is easily seen to be true for relations derived by cyclic permutation of 3 quantics, and 1.6, 2.6 and 3.3. Under the same, perhaps void, restriction the same relations hold for the values $[a]$ formed by $2n-2$

[8] The relation $[1:2, 3:4]+[1:3, 2:4]=1$ by which the $2N=6$ values of the ordinary cross ratio are shown to be functions of one of them—and which is equivalent to the additivity of linear measure—ceases to hold, for example, for squares of linear forms. Likewise, for general n, there hold special relations in the linear case, reducing the above sequence to that of squares 1, 4, 9, $\cdots$. A, B, and so on, and the effect of interchanging b_k and c_k, are cases of logarithmic antisymmetry.

[9] A for $n=3$ and the last quantic equal to 1 is the extension of Carnot's theorem given in my paper loc. cit. p. 976.

of $2n-1$ given hypersurfaces, the dimension of space having been diminished by 1.

3.7. *Variation.* $[a]$ remains unchanged if b_k is replaced by $b_k+\lambda b_jc_j$, or c_k by $c_k+\lambda b_jc_j$. Hence $[a]$ is determined if, in case $\bar{b}_k \geqq \bar{b}_j+\bar{c}_j$, instead of b_k, only the intersection of $b_k=0$ with $b_jc_j=0$ (and thereby also $\bar{b}_k$) is known. Cf. the end of 3.1, and of 3.5.

In some cases the cross ratio $[a_0, a]$ is quite independent of a_0 (of a given degree).[10] By 3.1, this happens if, and only if, all multiplicities of the points P_i of $[a]$ are equal to 0, that is, if every meet of n hypersurfaces, one of each pair, belongs to a further hypersurface, for example, in the linear combination case 3.8.

Already for $n=2$ this is not the only case of constant $[1, a]$. Let b_1, c_1, b_2, c_2—no two of which shall have a common factor of positive degree—have $p \geqq 0$ common points, and b_1, c_1, b_2 have γ_2 additional common points, and define β_1, γ_1, β_2 similarly. Then, of course, $\bar{b}_1\bar{c}_1 \geqq p+\beta_2+\gamma_2$ and 5 analogous inequalities hold, of which—by the before-said—4 are equations if, and only if, $[1, a]$ is constant. The 4 equations give $\prod \bar{a}=0$ (as stated before); let $\bar{a}_1=0$, so that $\bar{b}_1=\bar{c}_1$, $\beta_1=\gamma_1$, and $\gamma_2-\beta_2=\bar{b}_1\bar{a}_2$, which may be assumed not less than 0. $\bar{b}_1$, $\bar{c}_2$, $\bar{a}_2$, p, β_1, β_2 are connected by (1) $\bar{b}_1^2 \geqq p+2\beta_2+\bar{b}_1\bar{a}_2$, (2) $\bar{c}_2^2+\bar{c}_2\bar{a}_2 \geqq p+2\beta_1$, (3) $\bar{b}_1\bar{c}_2=p+\beta_1+\beta_2$. Eliminating p, there remains (4) $0, \bar{b}_1\bar{c}_2-\bar{a}_1\bar{c}_2-\bar{c}_2^2+\beta_1 \leqq \beta_2 \leqq \bar{b}_1\bar{c}_2-\beta_1, b_1^2-\bar{b}_1\bar{a}_2-\bar{b}_1\bar{c}_2+\beta_1$. Eliminating β_2, we have (5) $0, \bar{b}_1\bar{a}_2+\bar{b}_1\bar{c}_2-\bar{b}_1^2 \leqq \beta_1 \leqq \bar{b}_1\bar{c}_2, (\bar{a}_2+\bar{c}_2)\bar{c}_2/2$ and $(\bar{b}_1-\bar{c}_2)\bar{a}_2 \leqq (\bar{b}_1-\bar{c}_2)^2$,[11] that is, either $S1: \bar{a}_2 \leqq \bar{b}_1-\bar{c}_2$, which may be written $\bar{c}_2 \leqq \bar{b}_2 \leqq \bar{b}_1$, or $S2: \bar{a}_2 \leqq \bar{b}_1 \leqq \bar{c}_2$, which may be written $\bar{b}_2-\bar{c}_2 \leqq \bar{b}_1 \leqq \bar{c}_2 \leqq \bar{b}_2$ ($\bar{a}_2 \leqq \bar{b}_1$ follows by eliminating β_1, or from (1), except for $\bar{b}_1=\bar{c}_1=0$, a linear combination case). In both cases, the inequality (6) $\bar{b}_1\bar{a}_2+\bar{b}_1\bar{c}_2-\bar{b}_1^2 \leqq (\bar{a}_1\bar{c}_2+\bar{c}_2^2)/2$ is automatically fulfilled: for $S1$ this is obvious, for $S2$ we write (6) $(2\bar{b}_1-\bar{c}_2)\bar{a}_2 \leqq \bar{b}_1^2+(\bar{b}_1-\bar{c}_2)^2$, the left side, if greater than 0, being not greater than $(2\bar{b}_1-\bar{c}_2)\bar{b}_1$ not greater than the right side.

For given $\bar{b}_1$, $\bar{c}_2$, we now may choose $\bar{a}_2>0$ according to $S1$ or $S2$, or $S3: \bar{a}_2=0$, with $\bar{b}_1>\bar{c}_2$, or $S4: \bar{a}_2=0$, $\bar{b}_1=\bar{c}_2$; then β_1 according to (5), which in cases $S1$, $S3$, $S4$ reduces to $0 \leqq \beta_1 \leqq (\bar{a}_1+\bar{c}_2)\bar{c}_2/2$; then β_2 according to (4), which in case $S4$ reduces to $\beta_2=\beta_1>0$ (for $\beta_1=0$ we have a linear combination case); then p by (3), $\bar{b}_2$, γ_2. $S3$ and $S4$ are the situationally invariant cases. Arranging by ascending $\max(\bar{b}_1, \bar{c}_2)$, the table of "nonlinear-combination-cases" begins

[10] By 3.3 it is necessary and sufficient for this, that it be so for a linear form $a_0=1$. By 3.2 there must be $\prod \bar{a}=0$. If the cross ratio is 1, it is also independent of $\bar{a}_0$.

[11] Also immediately as $(1)+(2)-2(3)$.

S	1	2	2	1	1	3	3	2	2	4	4	2	2	2
$\bar{b}_1$	1	1	1	2	2	2	2	2	2	2	2	2	2	2
$\bar{c}_2$	0	1	2	0	0	1	1	1	1	2	2	2	2	2
$\bar{a}_2$	1	1	1	1	2	0	0	1	1	0	0	1	1	2
β_1	0	1	2	0	0	0	0	0	1	1	2	2	3	4
β_2	0	0	0	0	0	1	2	0	1	1	2	0	1	0
p	0	0	0	0	0	1	0	2	0	2	0	2	0	0
$\bar{b}_2$	1	2	3	1	2	1	1	2	2	2	2	3	3	4
γ_2	1	1	1	2	4	1	2	2	3	1	2	2	3	4
	A	B		C	D	E	F	G	H	I	J			

All 10 cases $A, \cdots, J$, containing only lines and conics, are realizable. Of the 4 situationally invariant cases E, F, I, J, the latter concerns 4 conics in the position described by the theorem:

Of 4 *given pairs of points every* 3 *pairs are on a conic if, and only if, either* (1) *all* 4 *pairs are on a conic, or* (2) *a point and a line harmonic to the* 4 *pairs exist.*

Indeed, let B_1, C_1, B_2, C_2 be pairs of points, and $B_1C_1B_2$ on a conic c_2, and so on. Choose $C_2=(I, J)$, then b_1, c_1, b_2 are circles. If they coincide, we have case (1); otherwise B_1, C_1, B_2 are pairs of inverse points for a certain circle O. If O is a line, we have case (2); otherwise, in the inversion at O, c_2 becomes a quartic c_2', meeting c_2 at B_1, C_1, B_2 and in 4 points of O, which is impossible.[12]

F (on a pair of conics intersecting in 4 points $ABCD$ and the pair of lines AB, CD) is a case of cross ratio 1, for the cross ratio is the same for all pairs of conics, as it cannot become 0 (or ∞), and equals 1 for coinciding conics; for two circles, F becomes the theorem of the radical axis. E concerns two conics as in F and the lines AB, AC; by a limiting case of F, either conic may be replaced by BC and its tangent at A, so the fixed cross ratio equals that of AB, AC and the two tangents. I yields a theorem on 4 circles, which by inversion becomes the statement that a variable circle is met by a fixed triangle and its circumcircle in a constant cross ratio.

3.8. *Linear combination.* 1.8 and 2.8 extend to the case where the unaltered a_k are rational functions.

[12] The non-existence of c_2, in this case, is also easily proved algebraically, or by remarking that the directions of its axes would have to be harmonic to any 2 of the 3 lines through O and B_1, C_1, B_2.

If $m+1>0$ quantics b_k are linear forms β_k of $m+1$ quantics d_k, and if the corresponding c_k are also linear forms γ_k of d_k, then by repeated application of 1.8, or 2.8, we have $[a]=[\alpha]^{\Pi_k \bar{d}}$, where $\alpha_k=\beta_k:\gamma_k$. In particular we mention that (a) two pairs of hypersurfaces belonging to a pencil are met by every line in the same cross ratio (and by any other curve in a power of that cross ratio)—a generalization of a fundamental theorem of projective geometry—and this cross ratio is a projective invariant; (b) n pairs of hyperplanes through a point are cut by every hyperplane in the same cross ratio.

The cross ratio of $n+1$ pairs of hyperplanes, or points, considered as function of one of them, is a quotient of two quantics of degree 2^{n-1}, and constant on a hypersurface of degree 2^{n-1}. For $n=2$, the cross ratio of 3 pairs of points, or lines, is 1 if, and only if, they are on a conic, which is the fact underlying the projective generation of conics. For general n, the cross ratio of $n+1$ points a and of the fundamental points is 1 if, and only if, the product of the coaxial minors of even degree of a equals that of those of odd degree. For $n=1$, this leads to $m=0$.

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