# ASYMPTOTIC DISTRIBUTION OF SUMS OF RADEMACHER FUNCTIONS

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1. **Introduction.** The central limit theorem of the calculus of probability applied to the special case of the Rademacher functions $r_k(t) = \text{sign} (\sin 2^k \pi t)$ can be stated as follows:

$$\lim_{n \to \infty} \left| E_t \left\{ \frac{1}{(2n)^{1/2}} \left| \sum_{k=1}^{n} r_k(t) \right| < a \right\} \right| = \frac{1}{\pi^{1/2}} \int_{-a}^{a} e^{-u^2} du.$$

This form of the theorem suggests two generalizations, the first of which consists in replacing the constant $a$ by a function $f(t)$. The second generalization is obtained by replacing the constant upper limit of summation $n$ by a sequence of integral-valued functions $N_n(t)$, so that the number of Rademacher functions in the sum varies with $t$. These are proved by combining the standard techniques of the calculus of probability with the methods of orthogonal series. Both proofs make use of the continuity theorem of Fourier-Stieltjes transforms:

*If* $f(t) = \lim_{n \to \infty} f_n(t)$, *and if*

$$\lim_{n \to \infty} \int_0^1 \exp [ixf_n(t)] dt = e^{-x^2/4}$$

*uniformly in $x$, then*

$$\lim_{n \to \infty} \left| E_t \left\{ | f_n(t) | < a \right\} \right| = \frac{1}{\pi^{1/2}} \int_{-a}^{a} e^{-u^2} du.$$

Essential use is made of the Walsh-Kaczmarz system[1] of orthonormal functions, which have two useful properties: (1) each of them is equal to a finite product of Rademacher functions, and (2) any Lebesgue-square integrable function can be expanded in a series of these functions, which series will converge in the mean with index two.

2. **First generalization.** The first theorem expresses in asymptotic form the measure of the point set over which the sum of the Rademacher functions is bounded by a given function.

[1] Cf. S. Kaczmarz and H. Steinhaus, *Le système orthogonal de M. Rademacher*, Studia Mathematica vol. 2 (1930) and J. L. Walsh, *A closed set of normal orthogonal functions*, Amer. J. Math. vol. 45 (1923) pp. 5–24.

THEOREM. *If $f(t)$ is a non-negative, measurable function in the interval $(0, 1)$, then*

$$\lim_{n \to \infty} \left| \, \underset{t}{E} \left\{ \frac{1}{(2n)^{1/2}} \left| \sum_{k=1}^{n} r_k(t) \right| < f(t) \right\} \right| = \frac{1}{\pi^{1/2}} \int_0^1 \int_{-f(t)}^{f(t)} e^{-u^2} du \, dt.$$

PROOF. Let $F_n(t)$ be $(2n)^{-1/2} \sum_{k=1}^{n} r_k(t)$ and let $S_r(t)$ be the partial sum of the expansion of a Lebesgue-square integrable function $h(t)$ in a series of Walsh-Kaczmarz functions. It follows from the first property of these functions that

$$\lim_{n \to \infty} \int_0^1 S_r(t) \exp \left[ i x F_n(t) \right] dt = e^{-x^2/4} \int_0^1 h(t) dt.$$

In view of the second property of the Walsh-Kaczmarz functions, this implies that

$$\lim_{n \to \infty} \int_0^1 h(t) \exp \left[ i x F_n(t) \right] dt = e^{-x^2/4} \int_0^1 h(t) dt.$$

If $h(t)$ be the characteristic function of an interval $E$, then

$$\lim_{n \to \infty} \int_E \exp \left[ i x F_n(t) \right] dt = |E| \cdot e^{-x^2/4}.$$

By virtue of the continuity theorem of Fourier-Stieltjes transforms, it can be demonstrated that

$$\lim_{n \to \infty} \left| \, \underset{t}{E} \left\{ t \in E; \, |F_n(t)| < a \right\} \right| = \frac{|E|}{\pi^{1/2}} \int_{-a}^{a} e^{-u^2} du.$$

Hence, if $g(t)$ be a non-negative step-function having the constant value $a_k$ over the interval $E_k$, then

$$\lim_{n \to \infty} \left| \, \underset{t}{E} \left\{ |F_n(t)| < g(t) \right\} \right| = \lim_{n \to \infty} \sum_{k=1}^{\infty} \left| \, \underset{t}{E} \left\{ t \in E_k; \, |F_n(t)| < a_k \right\} \right|.$$

Inspection shows that the limit on the right is equal to

$$\frac{1}{\pi^{1/2}} \int_0^1 \int_{-g(t)}^{g(t)} e^{-u^2} du \, dt.$$

The theorem now follows by considering $f(t)$ to be the limit of properly chosen step-functions.

3. **Second generalization.** In order to obtain the generalization of the central limit theorem in which the upper limit of summation of

the Rademacher functions is a given function, we must properly qualify this function.

THEOREM. *Let* $N_n(t) = nN(t) + Q_n(t)$, *where* $N_n(t)$ *and* $N(t)$ *assume as values only zero and the positive integers and where* $N(t)$ *is Lebesgue integrable, with* $p = \int_0^1 N(t)dt$. *If* $\lim_{n\to\infty} n^{-1/2} |Q_n(t)| = 0$, *then for any non-negative* $a$,

$$\lim_{n\to\infty} \left| \mathop{E}_t \left\{ \frac{1}{(2np)^{1/2}} \left| \sum_{k=1}^{N_n(t)} r_k(t) \right| < a \right\} \right| = \left(\frac{p}{\pi}\right)^{1/2} \sum_{M=0}^{\infty} \frac{|E_M|}{M^{1/2}} \int_{-a}^{a} e^{-px^2/M}dx$$

*where*

$$E_M = \mathop{E}_t \{ N(t) = M \}.$$

PROOF. Since the conditions of the theorem imply that

$$\sum_{k=1}^{N_n(t)} r_k(t) = \sum_{k=1}^{nN(t)} r_k(t) + S_n(t)$$

where $S_n(t)$ is such that $|S_n(t)| \leq |Q_n(t)|$, it follows that

$$\lim_{n\to\infty} \int_{E_M} \exp\left[ \frac{ix}{(2np)^{1/2}} \sum_{k=1}^{N_n(t)} r_k(t) \right] dt$$

$$= \lim_{n\to\infty} \int_{E_M} \exp\left[ \frac{ix}{(2np)^{1/2}} \sum_{k=1}^{nN(t)} r_k(t) \right] dt.$$

Since $N(t)$ is constant over $E_M$, the second limit equals $|E_M| \cdot e^{-Mx^2/4p}$. By virtue of the continuity theorem of Fourier-Stieltjes transforms, it follows that

$$\lim_{n\to\infty} \left| \mathop{E}_t \left\{ t \in E_M; \frac{1}{(2np)^{1/2}} \left| \sum_{k=1}^{N_n(t)} r_k(t) \right| < a \right\} \right|$$

$$= |E_M| \cdot \left(\frac{p}{M\pi}\right)^{1/2} \int_{-a}^{a} e^{-px^2/M}dx,$$

and hence the theorem is proved.

The conclusion of the theorem obviously holds under the stronger condition that there exists an $N(t)$ and an $A$ such that $|N_n(t) - nN(t)| < A$ for all $n$.

4. **Sums of cosines with big gaps.** By a similar method, using a Fourier series instead of a series of Walsh-Kaczmarz functions, it can be proved[2] that if $f(t)$ is a positive, measurable function in the inter-

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[2] Professor M. Kac has proved the central limit theorem for cosines with big gaps in *Note on power series with big gaps*, Amer. J. Math. vol. 61 (1939) pp. 473–476.

val $(0, 1)$ and if $\{n_k\}$ is a sequence of positive integers satisfying the gap condition: $\lim_{k \to \infty} n_k/n_{k-1} = \infty$, then

$$\lim_{m \to \infty} \left| \mathop{E}_{t} \left\{ \frac{1}{m^{1/2}} \left| \sum_{k=1}^{m} \cos 2\pi n_k t \right| < f(t) \right\} \right| = \frac{1}{\pi^{1/2}} \int_0^1 \int_{-f(t)}^{f(t)} e^{-u^2} du \, dt.$$

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