A NOTE ON THE ZEROS OF THE SECTIONS OF A PARTIAL FRACTION

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1. **Introduction.** If $f(z)$ is a rational function with a total of three distinct zeros and poles, the zeros of its logarithmic derivative may be located as points in the complex plane by aid of the following theorem.

THEOREM 1. *The zeros of the partial fraction*

$$F(z) = \frac{m_1}{z - z_1} + \frac{m_2}{z - z_2} + \frac{m_3}{z - z_3}, \qquad m_1 m_2 m_3 \neq 0,$$

where z_1, z_2 and z_3 are three distinct, noncollinear points, lie at the foci of the conic which touches the line segments (z_2, z_3), (z_3, z_1) and (z_1, z_2) in the points ζ_1, ζ_2 and ζ_3 that divide these segments in the ratio $m_2:m_3$, $m_3:m_1$, and $m_1:m_2$ respectively. If $n = m_1 + m_2 + m_3 \neq 0$, this conic is an ellipse or hyperbola according as $nm_1m_2m_3 > 0$ or < 0. If $n = 0$, the conic is a parabola whose axis is parallel to the line joining the origin to the point $\nu = m_1z_1 + m_2z_2 + m_3z_3$.

In the special case $m_1 = m_2 = m_3 = 1$, this theorem was proved geometrically by Bôcher and Grace.[1] In the general case it was first deduced by Linfield as a corollary to the following theorem which in turn was established by the use of line coordinates and polar forms.[2]

THEOREM 2. *The zeros of the partial fraction $F(z) = \sum_{j=1}^{p} m_j/(z - z_j)$ lie at the foci of the curve $C(z_1, z_2, \cdots, z_p; m_1, m_2, \cdots, m_p)$ of class $p-1$ which touches each of the $p(p-1)/2$ line-segments (z_j, z_k) in a point dividing it in the ratio $m_j:m_k$.*

In view, however, of the elementary character of Theorem 1, it

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[1] M. Bôcher, Ann. of Math. vol. 7 (1892) pp. 70–76; J. H. Grace, Proc. Cambridge Philos. Soc. vol. 11 (1901) pp. 352–357.

[2] For the case that all $m_j > 0$, see Siebeck, J. Reine Angew. Math. vol. 64 (1864) p. 175; M. Van den Berg, Niew Archief voor Wiskunde vol. 9 (1882) pp. 1–14, 60, vol. 11 (1884) pp. 153–186, vol. 15 (1899) pp. 100–164; J. Juhel-Renjoy, C. R. Acad. Sci. Paris vol. 142 (1906); P. J. Heawood, Quart. J. Math. vol. 38 (1907) pp. 84–107; and M. Fujiwara, Tôhoku Math. J. vol. 9 (1916) pp. 102–108. It is to be observed that, although priority for the theorem when all $m_j > 0$ is usually accorded to Van den Berg, it should rightfully be given to Siebeck. For arbitrary integral m_j, see B. Z. Linfield, Bull. Amer. Math. Soc. vol. 27 (1920) pp. 17–21 and Trans. Amer. Math. Soc. vol. 25 (1923) pp. 239–258.

seems desirable to furnish for it an elementary proof based upon some familiar property of the conics. In the next paragraph we shall construct such a proof, based upon the optical properties of the conics and upon some apparently new propositions concerning the relative positions of the zeros of the partial fraction $F(z)$, and those of its two sections, $F_1(z)=\sum_1^k m_j(z-z_j)^{-1}$ and $F_2(z)=\sum_{k+1}^p m_j(z-z_j)^{-1}$. These propositions, which will be derived in §§2 and 3 by the aid of very simple analysis, will also be used in §4 together with Theorem 2 to obtain some relations among the foci of the curves $C(z_1, z_2, \cdots, z_k; m_1, m_2, \cdots, m_k)$, $C(z_{k+1}, \cdots, z_p; m_{k+1}, \cdots, m_p)$ and $C(z_1, z_2, \cdots, z_p; m_1, m_2, \cdots, m_p)$. Finally, in §5, applications of the method will be indicated for nonlinear, partial fractions.

2. **Proof of Theorem 1.** Assuming $n\neq 0$ and $n_3=m_1+m_2\neq 0$, let us write

$$F(z) = \frac{n(z - Z_1)(z - Z_2)}{(z - z_1)(z - z_2)(z - z_3)}\,. \tag{2.1}$$

Since by definition

$$n_3\zeta_3 = m_2z_1 + m_1z_2, \tag{2.2}$$

it follows that ζ_3 is the zero of the partial fraction $F_1(z)=m_1/(z-z_1)+m_2/(z-z_2)$. Hence,

$$\frac{\zeta_3 - z_3}{n}F(\zeta_3) = \frac{m_3}{n} = \frac{(Z_1 - \zeta_3)(Z_2 - \zeta_3)}{(z_1 - \zeta_3)(z_2 - \zeta_3)},$$

$$\arg (Z_1 - \zeta_3)/(z_1 - \zeta_3) \equiv \arg (z_2 - \zeta_3)/(Z_2 - \zeta_3) \quad (\text{mod } 2\pi) \text{ if } nm_3 > 0, \tag{2.3}$$

$$\arg (Z_1 - \zeta_3)/(z_1 - \zeta_3) \equiv \arg (\zeta_3 - z_2)/(Z_2 - \zeta_3) \quad (\text{mod } 2\pi) \text{ if } nm_3 < 0. \tag{2.4}$$

By clearing equation (2.1) of fractions and equating the coefficients of z on both sides of the resulting equation, we learn that

$$(Z_1 + Z_2)/2 = (n_1z_1 + n_2z_2 + n_3z_3)/n$$

where $n_1=m_2+m_3$, $n_2=m_3+m_1$ and $n_3=m_1+m_2$. Since by hypothesis $n_3\neq 0$, the point $(Z_1+Z_2)/2$ cannot lie on the segment (z_1, z_2) and hence not both Z_1 and Z_2 may lie on this segment.

If ζ_3 is an interior division point of the line segment (z_1, z_2) and thus $m_1m_2>0$, we learn from equations (2.3) and (2.4) that the line (z_1, z_2) makes equal angles with lines (Z_1, ζ_3) and (Z_2, ζ_3) and does or does not separate the points Z_1 and Z_2 according as $nm_3<0$ or >0.

Likewise, if ζ_3 is an exterior division point of the line segment (z_1, z_2) and thus $m_1m_2<0$, we learn from (2.3) and (2.4) that the line (z_1, z_2) makes equal angles with the lines (Z_1, ζ_3) and (Z_2, ζ_3) and does or does not separate the points Z_1 and Z_2 according as $nm_3>0$ or <0. Hence, using the optical properties of the conics, we infer that at point ζ_3 the line (z_1, z_2) is tangent to a conic with foci at points Z_1 and Z_2 and that this conic is an ellipse or hyperbola according as $nm_1m_2m_3>0$ or <0.

Now, by merely permuting the subscripts 1, 2, and 3, we may complete the proof of the part of the theorem involving the ellipse and hyperbola.

To prove the part involving the parabola, let us, since $n=0$, write instead of (2.1) the expression

$$F(z) = \frac{\nu(z - Z_1)}{(z - z_1)(z - z_2)(z - z_3)} \cdot$$

The quantity $\nu=m_1z_1+m_2z_2+m_3z_3\neq 0$ because, by hypothesis, the points z_1, z_2 and z_3 are noncollinear. Since now ν replaces the factor $(z-Z_2)$, a repetition of the above details would prove at once that line (z_1, z_2) is tangent at point ζ_3 to the parabola described in Theorem 1.

Remark. It is obvious from the proof that Theorem 1 holds not only, as usually asserted, when the m_j are integers but also when they are arbitrary real numbers. Using this fact, we may extend our theorem to the case that, for example, $n_3=m_1+m_2=0$ but $n\neq 0$. For, taking n_3 very small, we find that $nm_3>0$, $m_1m_2<0$, and, hence, $nm_1m_2m_3<0$. The corresponding conic is therefore a hyperbola which is tangent at ζ_3 to the line (z_1, z_2). Now, as $n_3\to 0$, the point ζ_3 goes to infinity and, hence, the line (z_1, z_2) becomes an asymptote of the hyperbola.

3. **Sections of a linear partial fraction.** In order to generalize Theorem 1, let us now assume that $n=m_1+m_2+\cdots+m_p\neq 0$, $n_1=m_1+m_2+\cdots+m_k\neq 0$, $n_2=n-n_1\neq 0$.

Let us write

$$F(z) = \sum_{j=1}^{p} \frac{m_j}{z - z_j} = \frac{n(z - Z_1)(z - Z_2)\cdots(z - Z_{p-1})}{(z - z_1)(z - z_2)\cdots(z - z_p)},$$

$$F_1(z) = \sum_{j=1}^{k} \frac{m_j}{z - z_j} = \frac{n_1(z - z_1')(z - z_2')\cdots(z - z_{k-1}')}{(z - z_1)(z - z_2)\cdots(z - z_k)},$$

$$F_2(z) = \sum_{j=k+1}^{p} \frac{m_j}{z - z_j} = \frac{n_2(z - z_1'')(z - z_2'')\cdots(z - z_{p-k-1}'')}{(z - z_{k+1})(z - z_{k+2})\cdots(z - z_p)} \cdot$$

Since $F(z)=F_1(z)+F_2(z)$, it follows that

$$n(z-Z_1)\cdots(z-Z_{p-1})$$
$$= n_1(z-z_1')\cdots(z-z_{k-1}')(z-z_{k+1})\cdots(z-z_p)$$
$$+ n_2(z-z_1)\cdots(z-z_k)(z-z_1'')\cdots(z-z_{p-k-1}'').$$

Hence, if z'' is any z_j'' or a point z_j with $1\leqq j\leqq k$, and if Z is any point Z_j, we find that

$$(3.1)\qquad \frac{(z''-Z_1)\cdots(z''-Z_{k-1})(z''-Z_k)\cdots(z''-Z_{p-1})}{(z''-z_1')\cdots(z''-z_{k-1}')(z''-z_{k+1})\cdots(z''-z_p)}=\frac{n_1}{n},$$

$$(3.2)\qquad \frac{(Z-z_1')\cdots(Z-z_{k-1}')(Z-z_{k+1})(Z-z_{k+2})\cdots(Z-z_p)}{(Z-z_1)\cdots(Z-z_{k-1})(Z-z_k)(Z-z_1'')\cdots(Z-z_{p-k-1}'')}=-\frac{n_2}{n_1}.$$

Now, taking the moduli and amplitudes of equations (3.1) and (3.2), we derive the following theorem.

THEOREM 3. *Given the partial fraction $F(z)=\sum_1^p m_j(z-z_j)^{-1}$ and its sections $F_1(z)=\sum_1^k m_j(z-z_j)^{-1}$ and $F_2(z)=\sum_{k+1}^p m_j(z-z_j)^{-1}$, where $n_1=\sum_1^k m_j\neq 0$, $n_2=\sum_{k+1}^p m_j\neq 0$, and $n=n_1+n_2\neq 0$. Denote by S the set of $p-1$ zeros of $F(z)$; by S_1 the set of $p-1$ points comprised of the $k-1$ zeros of $F_1(z)$ and the $p-k$ points $z_{k+1}, z_{k+2}, \cdots, z_p$; and, by S_2, the set of $p-1$ points comprised of the k points $z_1, z_2, \cdots, z_k$ and the $p-k-1$ zeros of $F_2(z)$.*

A. *Then, if z'' is any point of S_2, the product of its distances to the points of S is $|n_1/n|$ times the product of its distances to the points of S_1. Also, if each point in S is paired in any manner with a point in S_1, the sum of the angles subtended at z'' by the $p-1$ pairs of points is zero or π* (mod 2π) *according as $n_1 n>0$ or <0.*

B. *Further, if Z is any point of S, the product of its distances to the points of S_1 is $|n_2/n_1|$ times the product of its distances to the points in S_2. Also, if each point in S_1 is paired in any manner with a point in S_2, the sum of the angles subtended at Z by the $p-1$ pairs of points is zero or π* (mod 2π) *according as $n_1n_2<0$ or >0.*[3]

For example, if $p=4$ and $k=2$, then $F_1(z)$ has as its only zero the point z' dividing the line segment (z_1, z_2) in the ratio $m_1:m_2$ and $F_2(z)$ has as its only zero the point z'' dividing the line segment (z_3, z_4) in

[3] In the case $m_1=m_2=\cdots=m_p=k=1$, the set S_2 consists of the point z_1 and the $p-2$ zeros of $F_2(z)$. Bôcher (see footnote 1) states without proof Theorem 3, but takes the point z'' to be only the point z_1, making no mention of the fact that z'' may be also any of the other $p-2$ points of S_2. In Theorem 3, the sense in which an angle subtended by a line segment is to be measured may be determined from formulas (3.1) and (3.2).

the ratio $m_3:m_4$. Using part B of Theorem 3, we may prove that the sum of the angles subtended in any zero Z of $F(z)$ by the line segments (z', z_1) and (z_4, z'') is the supplement of the angle subtended in Z by the segment (z_2, z_3). Likewise, by using part A, we may prove that, if ψ denotes double the angle from the line (z_3, z_4) to the line (z_1, z_2), then $(\psi-\pi)$ equals the sum of the angles subtended by the line segment (z', z'') in the zeros Z_1, Z_2, and Z_3 of $F(z)$ (mod 2π).

4. **Geometric interpretations of Theorem 3.** First, Theorem 3 may be restated in terms of two types of generalizations of the circle which were studied by Darboux.[4] One is the so-called lemniscate, the locus of a point which moves so that the product of its distances from one set of fixed points (called poles) is a constant multiple of the product of its distances from another set of fixed points (also called poles). The other is the so-called stelloid, the locus of a point which moves so that the sum of the angles subtended in it by a set of pairs of fixed points (called poles) is constant. These two families are respectively the equipotential curves and lines of force in the field due to unit attractive particles at the points of the one set and unit repulsive particles at the points of the other set. Thus, according to Theorem 3, *the points of S_2 lie at the intersections of a certain lemniscate and a certain stelloid, both of which have the points of S and S_1 as poles*, whereas *the points of S lie at the intersections of a lemniscate and a stelloid, both of which have the points of S_1 and S_2 as poles.*

Secondly, Theorem 3 may be used in combination with Theorem 2. By Theorem 2, the points $Z_1, Z_2, \cdots, Z_{p-1}$ are the foci of the curve $C_0 \equiv C(z_1, \cdots, z_p; m_1, \cdots, m_p)$ of class $p-1$; the points $z_1', z_2', \cdots, z_{k-1}'$ the foci of the curve $C_1 \equiv C(z_1, \cdots, z_k; m_1, \cdots, m_k)$ of class $k-1$; and the points $z_1'', z_2'', \cdots, z_{p-k-1}''$ the foci of the curve $C_2 \equiv C(z_{k+1}, \cdots, z_p; m_{k+1}, \cdots, m_p)$ of class $p-k-1$. Thus, if $p=6$ and $k=3$ and all $m_j>0$, the points z_1' and z_2' are the foci of the ellipse which touches the line segments (z_1, z_2), (z_2, z_3) and (z_3, z_1) in points dividing these segments in the respective ratios $m_1:m_2$, $m_2:m_3$ and $m_3:m_1$, while the points z_1'' and z_2'' are the foci of the ellipse which touches the line segments (z_4, z_5), (z_5, z_6) and (z_6, z_4) in the points dividing these segments in the respective ratios $m_4:m_5$, $m_5:m_6$ and $m_6:m_4$.

The set S of Theorem 3 consists thus of the foci of the curve C_0. The set S_1 consists thus of the points $z_{k+1}, z_{k+2}, \cdots, z_p$ and the foci of the curve C_1 or, what amounts to the same, the foci of the degen-

[4] G. Darboux, *Sur une classe remarquable de courbes et de surfaces algebrique*, Paris, 1873, pp. 66–73.

erate curve $C' \equiv C(z_1, \cdots, z_k, z_{k+1}, \cdots, z_p; m_1, \cdots, m_k, 0, \cdots, 0)$ of class $p-1$. Similarly the set S_2 may be regarded as consisting of the foci of the degenerate curve $C''=C(z_1, \cdots, z_k, z_{k+1}, \cdots, z_p; 0, \cdots, 0, m_{k+1}, \cdots, m_p)$, also of class $p-1$. Thus, *Theorem* 3 *expresses two geometric relations regarding the foci of the three curves* C_0, C' and C'' *of class* $p-1$.

5. **Further generalizations.** It is obvious that Theorem 3 may be extended at once to the more general partial fractions

$$F(z) = \sum_{j=1}^{p} \frac{m_j(z-a_{j1}) \cdots (z-a_{jr})}{(z-b_{j1}) \cdots (z-b_{js})}.$$

For example, let us take all $m_j > 0$, and

$$F(z) = \sum_{j=1}^{3} \frac{m_j}{(z-z_j)^q}, \qquad F_1(z) = \frac{m_1}{(z-z_1)^q}, \qquad F_2(z) = F(z) - F_1(z).$$

The zeros of $F_2(z)$ are the roots of the equations

$$z - z_2/z - z_1 = (m_2/m_1)^{1/q} \exp\left[\pi(2k+1)i/q\right],$$

where $k=0, 1, 2, \cdots, q-1$. These zeros may therefore be located as the intersections of the circle $|z-z_2| = (m_2/m_1)^{1/q}|z-z_1|$ with the q arcs of circles $\arg (z-z_2)/(z-z_1) = \pi(2k+1)/q$, $k=0, 1, 2, \cdots, q-1$. On the other hand, if ζ is any zero of $F_2(z)$,

$$\frac{(\zeta - z_1)^q}{n} F(\zeta) = \frac{m_1}{n} = \frac{(\zeta - Z_1) \cdots (\zeta - Z_{2q})}{(\zeta - z_2)^q(\zeta - z_3)^q}, \tag{5.1}$$

where $Z_1, Z_2, \cdots, Z_{2q}$ are the zeros of $F(z)$ and $n = m_1+m_2+m_3$. From equation (5.1) we may thus conclude that *the sum of the angles subtended in* ζ *by the segments from* z_2 *to any* q *of the points* Z_j *equals* (modulo 2π) *the sum of the angles subtended in* ζ *by the segments from the remaining* Z_j to z_3.

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