§16) to show that intuitionistic number theory admits, besides the extension which gives classical number theory, also an intuitionistic extension. Both extensions are simply consistent if the unextended system is, and the two extensions are contradictory to each other. The work involves formalization of the theory of certain primitive recursive predicates $R_k(e, x_1, \cdots, x_k, y)$ which in a formal intuitionistic system afford a representation of the theory of general and partial recursive functions and predicates. The present results combine with reasoning of Kleene (Abstract 48-1-85) to establish the independence of certain formulas of the intuitionistic predicate calculus, in particular of the formula $\neg\neg(x)(A(x) \vee \neg A(x))$. (Received August 8, 1945.)

## Statistics and Probability

### 241. Reinhold Baer: *Sampling from a changing population.*

The stochastic limits of certain functions of random samples are determined where the samples are taken from different distributions belonging to a continuous family of distributions. (Received August 22, 1945.)

### 242. J. L. Doob: *Markoff chains—denumerable case.*

Let $p_{ij}(t)$, $i, j=1, 2, \cdots, 0 \leqq t < \infty$, be the transition probability functions of a Markoff process. Let $x(t)$ be the (integral) value assumed by the probability system at time $t$. Necessary and sufficient conditions are found that the $p_{ij}(t)$ satisfy the systems of first order differential equations (*) $p'_{ik}(t) = -q_i p_{ik}(t) + \sum_{j \neq i} q_{ij} p_{jk}(t)$, $p'_{ik}(t) = -p_{ik}(t) q_k + \sum_{j \neq i} p_{ij}(t) q_{jk}$, where $q_i = -p'_{ii}(0)$, $q_{ik} = p'_{ik}(0)$ $(i \neq k)$. A detailed analysis is made of the processes for which the discontinuities of $x(t)$ are well ordered. It is shown that if $q_i$, $q_{ik}$ are specified arbitrarily except that $q_{ik} \geqq 0$, $q_i = \sum_j q_{ij}$, there is always a corresponding set of functions $\{p_{ij}(t)\}$ determining a Markoff process, but that in general there will be infinitely many such sets of functions, and even infinitely many satisfying (*), such that the discontinuities of $x(t)$ are well ordered. The initial conditions $p_{ij}(0) = \delta_{ij}$ are thus insufficient to determine uniquely the solutions to (*). (Received September 22, 1945.)

### 243. Mark Kac: *On the average of a certain Wiener functional.*

Let $x(t)$ be an element of the Wiener space. It is shown that the average of the functional $\exp\left(-z\int_0^1 |x(t)| dt\right)$ $(z>0)$ is given by the formula $\sum_1^\infty \kappa_j \exp\left(-(\delta_j/2)z^{2/3}\right)$, where $\delta_1, \delta_2, \cdots$ are positive zeros of the derivative of $P(y) = (2y)^{1/2}\{J_{-1/3}((2^{3/2}/3)y^{3/2}) + J_{1/3}((2^{3/2}/3)y^{3/2})\}$ and $\kappa_j = (1+\int_0^{\delta_j} P(y)dy)/\delta_j P(\delta_j)$. A related limit theorem in calculus of probability is discussed. In the course of the proof the following seemingly new result was also obtained: If $r_j$ is the $j$th positive root of $J_\nu(x)$ $(\nu \geqq 0)$ then $r_j^2 > \nu^2 + (2\pi j)^{2/3}\nu^{4/3}$. For $j$ fixed the estimate is weaker than a known asymptotic formula. The value of the estimate is due to the fact that $j$ can depend on $\nu$. (Received August 2, 1945.)

### 244. Isaac Opatowski: *Calculation of Markoff chains by incomplete gamma and beta functions and by Charlier polynomials.*

Several types of stochastic processes consisting of successive transitions between $n$ states $\{i\}_1^n$ are considered (cf. Bull. Amer. Math. Soc. vol. 51 (1945) p. 665). Call $P_{r,s}(t)$ the probability of being a system at the time $t$ in the state $s$ if it is at $t=0$ in the state $r$. Let the only transitions possible during $dt$ be $(i-1 \to i)(i+1 \to i)$ and the corresponding probabilities be equal respectively to $k_i dt+o(dt)$ and $g_i dt+o(dt)$ where $k_{n+1}\geqq 0$, the remaining constants $k_i$ being all greater than 0 and either independent of $i$ or forming an arithmetic progression. Let $g_i$ be either 0 or a constant $g>0$. If $g_m=0$ for a certain $m\leqq n-2$ then $P_{1,n}(t)$ is the convolution of $P_{1,m}(t)$ and $P_{m+1,n}(t)$. This reduces the calculation of $P_{1,n}(t)$ to the case when all $g_i$'s are not equal to 0, and $P_{1,n}(t)$ is obtained in this case as a power series in $g$ whose coefficients are expressed in terms of the functions specified in the title of this abstract. The paper is a part of two articles to be published in vol. 8 (1946) of the Bulletin of Mathematical Biophysics. (Received October 1, 1945.)

## Topology

### 245. R. H. Bing: *Collections cutting the plane.*

The following is proved for the plane: Suppose that $K$ is a bounded set of $n$ components which does not cut the point $A$ from the point $B$, that $G$ is a collection of compact continua with a closed sum, that the sum of no $n$ elements of $G$ cuts $A$ from $B$ in the complement of $K$ and that no two elements of $G$ intersect each other in the complement of $\overline{K}$ but each element of $G$ intersects $\overline{K}$. Then the sum of the elements of $G$ does not cut $A$ from $B$ in the complement of $K$. Also, there is a subset $T$ of $\overline{K}-K$ irreducible with respect to $\overline{K}-T$ not cutting $A$ from $B$ and such that if the components of $T$ are regarded as points, there is an arc (or a point) in the complement of $\overline{K}-T$ that contains $A$ and $B$ but no point of an element of $G$. (Received September 4, 1945.)

### 246. D. G. Bourgin: *Quadratic forms.*

This note gives a topological interpretation of the two numerical invariants characterizing a real quadratic form, namely the rank and the signature. Thus for $Q$ (not definite) in $n+1$ essential variables the signature is $n+2-\sum_0^n R_j$ where $R_j$ is the $j$-dimensional mod 2 Betti number of the configuration $Q=0$ in $(n+1)$-dimensional projective space. The case of definite forms is included by either identifying rank and signature here or by using $Q'=Q+x_{-1}^2-x_{-2}^2$. The appearance of the sum rather than the alternating sum of the Betti numbers is interesting. The invariant above may be taken as a definition of the signature for more general forms, and presumably other numerical invariants needed for more general forms may be introduced in a natural way by taking account of the topological aspects of the manifolds corresponding to $Q=0$. (Received September 22, 1945.)

### 247. W. H. Gottschalk: *Properties of minimal sets.*

Let $X$ be a topological space in which there operates a *flux*, that is, a homeomorphism, a mapping, a one-parameter group of transformations, or a one-parameter semi-group. A subset of $X$ is *minimal* in the sense of G. D. Birkhoff provided that it is a smallest orbit-closure. It is shown that a minimal-set partition carries over from a flux to a sub-flux. Minimality is characterized. The existence of minimal sets is demonstrated in case the phase space $X$ is compact. Finally, it is proved that if $X$ is a compact Hausdorff space, if $f$ is a continuous flux, and if $X$ is minimal under $f$, then either $X$ is minimal under every sub-flux or $X$ is the cartesian product, with bases properly identified, of the unit interval and a set minimal under some sub-flux. It is to be noted that minimal sets and almost periodicity are intimately related. (Received August 8, 1945.)