tive ring without nilpotent elements is a field. The purpose of the present paper is to make some further characterizations of subdirectly irreducible commutative rings. Let R be a commutative ring, not all elements of R being divisors of zero, and denote by D the set of all elements of R which are divisors of zero. Then R is subdirectly irreducible if, and only if, it has the following four properties: (i) the set of all elements x of R such that $Dx=0$ is a principal ideal $J=(j)\neq 0$, (ii) the set of all elements y of R such that $Jy=0$ is precisely D, (iii) R/D is a field, (iv) if d_1 is any element of D, not in J, there exists an element d_2 of D, not in J, such that $d_1d_2=j$. Some additional related results are also obtained. (Received February 20, 1945.)

81. Kathryn A. Morgan: *Representation of a positive binary form by a positive quaternary form.*

The conditions for representing a positive binary form as a sum of squares of linear forms were discussed by Mordell (Quart. J. Math. Oxford Ser. vol. 1 (1930) pp. 276–288) and Chao Ko (Quart. J. Math. Oxford Ser. vol. 8 (1937) pp. 81–98). This paper presents a method for finding the representation of a positive primitive binary form by a positive quaternary form and especially the number of representations of a primitive binary form as the sum of four squares. (Received March 3, 1945.)

82. F. E. Satterthwaite: *Error control in matrix calculation.* II.

The arithmetic calculation of the inverse of a matrix or of the solution of a set of simultaneous equations is often complicated by a serious magnification of rounding errors. The proposed method is as follows: (1) Each equation (or line) of the Doolittle solution is expressed approximately as an exact linear function of the original equations. (2) The discrepancy between the approximation and the ideal is adjusted by the same type of process as is used to adjust the original equations in a standard Doolittle method. (3) If the approximation is close enough, the coefficients in the adjustment will be small enough so that they can not cause any significant carrying forward of errors. A second approximation is sometimes necessary to satisfy this condition. The advantages of this method are: (1) It works for any matrix or set of equations. (2) It does not require an advance approximate solution. (3) Any number of decimal places may be carried with complete assurance that errors are never greater than one or two in the last decimal place. (4) Each step is self-checking. (5) The method is ideally suited for use with modern high speed calculating machines. (6) The routine can be easily taught to the average clerk. (Received February 5, 1945.)

83. J. E. Wilkins: *A generalization of the Euler ϕ-function.*

One defines the function $\phi_n(k)$ so that exactly $\phi_n(k)$ of the k arithmetic progressions $mk+l$ ($m=0, 1, \cdots; l=0, 1, \cdots, k-1$) contain infinitely many numbers not divisible by an nth power greater than 1. It is shown that $\phi_n(k)$ is that multiplicative function for which $\phi_n(p^r)=p^r$ if $r<n$ and p is prime and $\phi_n(p^r)=p^r-p^{r-n}$ if $r\geqq n$ and p is prime. Thus $\phi_1(k)$ is the Euler ϕ-function. It is also shown that the function $\phi_n(k)/k$ is uniformly almost periodic in the sense of Bohr for $n\geqq 2$, and its asymptotic mean value is $1/\zeta(2n)$. For $n=2$, these results are due to Haviland, whose discussion, however, is not free from errors. (Received March 20, 1945.)

ANALYSIS

84. R. P. Agnew: *Spans of translations of peak functions.*

A peak function $F(x)$ is defined, in terms of positive constants a and b, by

$F(x)=a(b-|x|)$ when $|x|\leq b$ and $F(x)=0$ when $|x|>b$. It is shown that, for each $p>1$, the closure in the Lebesgue space L_p of the linear manifold determined by the translations of a peak function is the whole space L_p. Explicit formulas for approximations are given. Uniform approximation and truncated step functions are treated. (Received March 9, 1945.)

85. R. P. Boas: *Fundamental sets of analytic functions.*

A set $\{u_n(z)\}$ of functions analytic in $|z|<1$ is called fundamental if each function analytic in $|z|<1$ can be uniformly approximated in each circle $|z|\leq r<1$ by a linear combination of the $u_n(z)$. If $F(z)$ is entire, conditions on $F(z)$ and on the sequence $\{a_n\}$ which make the set $\{F(a_nz)\}$ fundamental have been given by Gelfond (Rec. Math. (Mat. Sbornik) N.S. vol. 4 (1938) pp. 149–156) and Markouchevitch (C.R. (Doklady) Acad. Sci. URSS vol. 43 (1944) pp. 3–6). In this paper it is shown that $\{F(a_nz)\}$ is fundamental if $F^{(n)}(0)\neq 0$ $(n=0, 1, 2, \cdots)$, and if when $F(z)$ is of order ρ and type τ, $\{a_n\}$ satisfies one of the following: (1) lim inf $n/|a_n|^\rho>\rho\tau$, (2) lim sup $n/|a_n|^\rho>e\rho\tau$. (1) improves a result of Gelfond; (2) is a result of Markouchevitch. It is further shown that (1) and (2) are best possible for all ρ. (Received March 15, 1945.)

86. Glenn James: *Certain general polynomial expansions.*

This paper develops an expansion of differentiable functions by means of a general operator, $\prod_{i=1}^{n}(I+D/(m-i+1))$. Special interpretations of D and m result in Taylor's theorem for one, and for several variables. The coefficients in this general expansion are, in general, functions of the number of terms. The resulting infinite series is shown to converge in instances where the corresponding Taylor's series diverges. (Received March 6, 1945.)

87. D. H. Lehmer: *On the Graeffe process for power series.*

The Graeffe root-squaring method, originally devised for the solution of algebraic equations, has been extended to cope with the problem of determining the absolute values of the zeros of entire functions (Polya: Zeitschrift für Mathematik Physik vol. 63 (1915) pp. 275–290, Ostrowski: Acta Math. vol. 72 (1940) pp. 99–257). The problem of the determination of the arguments of the complex zeros is considered in the present paper. A method is given which not only determines approximately each complex root but also affords a method of verification different from the forthright one of substituting into the proposed equation. A discussion is given of certain practical difficulties peculiar to power series. The method is illustrated in the case of the difficult Bessel function equation of F. Wolf: $J_0(z)J_2(z)=J_1^2(z)$. (Received March 5, 1945.)

88. Morris Marden: *A note on the zeros of the sections of a partial fraction.*

The following theorem was proved by Bôcher and Grace in the case $m_1=m_2=m_3=1$ and by Linfield in the general case: The zeros of the logarithmic derivative of the rational function $f(z)=(z-z_1)^{m_1}(z-z_2)^{m_2}(z-z_3)^{m_3}$ lie at the foci of the conic which touches the line-segments (z_1, z_2), (z_2, z_3) and (z_3, z_1) in points dividing these segments in the ratios $m_2:m_1$, $m_3:m_2$ and $m_1:m_3$ respectively. In the present note this theorem is given an elementary proof based upon the optical properties of conics and upon a relation between the zeros of the partial fraction $F(z)=\sum_1^p m_j(z-z_j)^{-1}$ and its sections

$F_1(z)=\sum_1^k m_j(z-z_j)^{-1}$ and $F_2(z)=\sum_{k-1}^p m_j(z-z_j)^{-1}$ when $p=3$ and $k=2$. The latter relation is then generalized to the case of arbitrary p and k and connected with the theorem due to Siebeck, Van den Berg, Heawood, Juhel-Renjoy and Linfield that the zeros of the logarithmic derivative of the rational function $f(z)=(z-z_1)^{m_1}(z-z_2)^{m_2} \cdots (z-z_p)^{m_p}$ lie at the foci of the curve of class $p-1$ which touches each of the $p(p-1)/2$ line-segments (z_j, z_k) in a point dividing this segment in the ratio $m_k:m_j$. (Received March 16, 1945.)

89. R. M. Robinson: *Univalent majorants.*

$g(z) \prec G(z)$ in a circle about the origin if both functions are regular there and $G(z)$ is univalent, if $g(0)=G(0)$, and if the set of values assumed by $g(z)$ is included in the set of values assumed by $G(z)$. It is shown that if $g(z) \prec G(z)$ for $|z|<1$ then $zg'(z) \prec zG'(z)$ for $|z| \leqq 3-2^{3/2}$. For any given $G(z)$ there is a largest r such that $g(z) \prec G(z)$ for $|z|<1$ implies $zg'(z) \prec zG'(z)$ for $|z| \leqq r$; this r assumes its smallest value $3-2^{3/2}$ for $G(z)=z/(1-z)^2$ and its largest value $2^{1/2}-1$ for $G(z)=z$. The passage from $g(z) \prec G(z)$ to $(zg(z))' \prec (zG(z))'$ is also considered, as well as the two inverse operations. The possibility of deriving one of the relations $f(z)/z \prec F(z)/z$, $f'(z) \prec F'(z)$, $zf'(z)/f(z) \prec zF'(z)/F(z)$ from another is studied, particular attention being paid to the case in which the given majorant is $(1+z)/(1-z)$, so that the majorized function has a positive real part. Typical results are: If $f(z)=z+\cdots$ is regular for $|z|<1$ and $\Re f'(z)>0$, then $f(z)/z \prec (2/z)\log(1-z)^{-1}-1$ for $|z|<1$. If $f(z)=z+\cdots$ is regular for $|z|<1$ and $\Re[zf'(z)/f(z)]>0$ (star mapping), then $f'(z) \prec (1+z)/(1-z)^3$ at least for $|z| \leqq (5-17^{1/2})/2$ (which extends a theorem of A. Marx). (Received March 7, 1945.)

90. A. R. Schweitzer: *Functional relations valid in the domains of abstract groups and Grassmann's space analysis.* III.

The author extends a type of generalized algebra of logic to $(n+1)$-ary composition $(n=1, 2, 3, \cdots)$ by defining a special case of the following postulates considered in a previous paper (abstract 51-1-20): 1. $f(u_1, u_2, \cdots, u_{n+1})=f(u, t_1, t_2, \cdots, t_n)$ where $u_i=f(x_i, t_1, t_2, \cdots, t_n)$ and $u=f(x_1, x_2, \cdots, x_{n+1})$. 2. $f(v_0, v_1, v_2, \cdots, v_n)=\phi(v, t_1, t_2, \cdots, t_n)$ where $v_0=\phi(x, t_1, t_2, \cdots, t_n)$, $v_i=\phi(t_i, t_1, t_2, \cdots, t_n)$ and $v=f(x, t_1, t_2, \cdots, t_n)$. 3. $f(x, x, \cdots, x)=x$. 4. Duals, in f and ϕ, of the prceeding postulates. 5. Closure of the set of elements x_i under the compositions f and ϕ. The special case noted is obtained by assuming (in postulate 2): 6. If $n>1$, then $v_1=v_2=\cdots=v_n$. 7. $f(v_0, v_1, v_2, \cdots, v_n)=\phi(t_1, t_1, t_2, \cdots, t_n)$. 8. Duals, in f and ϕ, of postulates 6 and 7. Then for $n=1$ under postulates 1–8, $f\{\phi(x, t), t\}=\phi\{f(x, t), t\}=t$. Also if $x_2=x_3=\cdots=x_{n+1}$ in $f(x_1, x_2, \cdots, x_{n+1})$ and $\phi(x_1, x_2, \cdots, x_{n+1})$ then postulates 1–5 become abstractly identical with those obtained from the latter for $n=1$. (Received March 23, 1945.)

91. Otto Szász: *On some summability methods with triangular matrix.*

In a recent paper (Ann. of Math. vol. 43) the author associated with a given series or sequence transform a class of triangular matrix transforms and discussed the regularity conditions of such transforms which correspond to some standard summability methods such as Abel summability. He now establishes for some of these methods the regularity conditions relative to Cesàro summability of a given order. As a result triangular transforms are found which are more powerful than the Cesàro scale. Such is the case, for example, for the series to sequence transform: $A_n=\sum_0^n u_\nu x_n^\nu$, where $x_n\uparrow 1$ as $n\to\infty$ so slowly that $n^r x_n^n\to 0$ for all integers r. (Received March 17, 1945.)

92. F. T. Wang: *Strong summability of Fourier series.*

Let $S_n(x)$ be the partial sum of the Fourier series $f(t)\sim a_0/2+\sum_{n=1}^\infty(a_n \cos nt + b_n \sin nt)$ at $t=x$, and let $\phi(t)=(1/2)\{f(x+t)+f(x-t)-2S\}$. The following result gives the solution of a problem proposed by Hardy and Littlewood (Fund. Math. vol. 25 (1935)): *If* $\int_0^t|\phi(u)|\{1+\log^+|\phi(u)|\}du=o(t)$ as $t\to 0$, then $\sum_{\nu=0}^n|S_\nu(x)-S|^2=o(n)$ as $n\to\infty$. (Received March 12, 1945.)

93. F. T. Wang: *Tauberian theorem of oscillating series.*

Let $\sigma_n^{(r)}$ be the rth Cesàro mean of the series $\sum_{n=0}^\infty a_n$; and $\sigma_n^{(r)}-s=o(n^{(\delta-1)})$ as $n\to\infty$ for $r>0$, $0<\delta<1$; and $a_n>-Kn^{-\delta}$; then the series $\sum_{n=0}^\infty a_n$ converges to S. An example shows that the order in the above inequality is the best possible in its kind. (Received March 12, 1945.)

94. H. J. Zimmerberg: *A class of definite boundary value problems.*

This paper is concerned with an extension of the results of Reid (Trans. Amer. Math. vol. 52 (1942) pp. 381–425) to differential systems consisting of the vector differential equation $y'=A(x)y+\lambda B(x)y$, and the two-point boundary conditions $(M_0+\lambda M_1)y(a)+(N_0+\lambda N_1)y(b)=0$, in which the elements of the coefficient matrices of the system are allowed to be complex-valued. It is shown that under suitable assumptions of definiteness such systems possess fundamental properties similar to those previously established for real-valued, definitely self-adjoint problems by Bobonis (doctoral dissertation, Chicago, 1939; *Contributions to the calculus of variations*, 1938–1941, pp. 99–138). In particular, this study yields new results for the definitely self-adjoint systems considered by Bobonis. It is also shown that certain important types of boundary value problems associated with the second variation of an isoperimetric problem of Bolza in the calculus of variations which are not definitely self-conjugate adjoint do belong to this new class of problems. (Received March 19, 1945.)

Geometry

95. L. K. Hua: *Geometries of matrices.* I. *Generalizations of von Staudt's theorem.*

A geometry is studied whose points are defined as the symmetric matrices Z of degree n; a class of points at infinity is to be added. The group of transformations of this geometry consists of all mappings $Z=(AZ^*+B)(CZ^*+D)^{-1}$ where the matrices A, B of degree n form the upper half of a symplectic matrix S of degree $2n$ while C, D form the lower half of S. The question of equivalence of systems of points with regard to this group is investigated. A generalization of von Staudt's theorem is obtained. Finally, some other geometries of a similar nature are discussed. (Received February 10, 1945.)

96. L. K. Hua: *Geometries of matrices.* I_1. *Arithmetical construction.*

The paper forms an illustration and a supplement to the first part. The geometry of two-rowed matrices is studied in more detail. It is shown that one of the conditions appearing in the generalization of von Staudt's theorem is redundant. (Received February 10, 1945.)