ON THE DEGREE OF APPROXIMATION OF FUNCTIONS BY FEJÉR MEANS

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1. **Continuous functions.** It has been proved by S. Bernstein that if $f(x)$ is periodic and of the class Lip α, $0<\alpha<1$, then the $(C, 1)$ means $\sigma_n(x)=\sigma_n(x;f)$ of the Fourier series of f satisfy the condition

$$(1.1)\qquad \sigma_n(x) - f(x) = O(n^{-\alpha}),$$

uniformly in x. The result is false for $\alpha=1$. The place of (1.1) is then taken by

$$(1.2)\qquad \sigma_n(x) - f(x) = O(\log n/n),$$

and, as simple examples show, the factor $\log n$ on the right cannot be removed (see, for example, A. Zygmund, *Trigonometrical series*, p. 62). It will be shown here that for power series the inequality (1.1) holds even for $\alpha=1$. More generally, we have the following theorem.

THEOREM 1. *Suppose that $f(x)$ is periodic, continuous, and that the Fourier series of f is of power series type,*

$$f(x) \sim \sum_{\nu=0}^{\infty} c_\nu e^{i\nu x}.$$

Then

$$(1.3)\qquad \left| \sigma_{n-1}(x) - f(x) \right| \leqq A\omega(2\pi/n),$$

where $\omega(\delta)$ is the modulus of continuity of f and A is an absolute constant.

The proof is based on the following lemma.

LEMMA. *Suppose that*

$$(1.4)\qquad g(x) \sim \sum_{-\infty}^{+\infty} \gamma_\nu e^{i\nu x}$$

satisfies $|g(x+h)-g(x)| \leqq M|h|$. Then

$$(1.5)\qquad \left| \tilde{\sigma}_{n-1}(x) - \tilde{g}(x) \right| \leqq BM/n,$$

where $\tilde{g}(x)$ is the function conjugate to $g(x)$ and $\tilde{\sigma}_n(x)$ are the $(C, 1)$ means of the series conjugate to (1.4).

For the proof of the lemma we note that

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$$\tilde{g}(x) = -\frac{1}{\pi}\int_0^\pi [g(x+t) - g(x-t)]\,\frac{1}{2}\cot\frac{1}{2}t\,dt,$$

$$\tilde{\sigma}_{n-1}(x) = -\frac{1}{\pi}\int_0^\pi [g(x+t) - g(x-t)] \cdot\left[\frac{1}{2}\cot\frac{1}{2}t - \frac{\sin nt}{n(2\sin(t/2))^2}\right]dt,$$

$$\tilde{g}(x) - \tilde{\sigma}_{n-1}(x) = \frac{1}{\pi}\int_0^\pi [g(x+t) - g(x-t)]\,\frac{\sin nt}{n(2\sin(t/2))^2}\,dt$$
$$= \frac{1}{\pi}\int_0^{\pi/n} + \frac{1}{\pi}\int_{\pi/n}^{\pi} = P_n + Q_n,$$

say. Since $|\sin nt| \leqq n\sin t \leqq n(2\sin(t/2))$ for $0 \leqq t \leqq \pi$,

$$|P_n| \leqq \frac{1}{\pi}\int_0^{\pi/n} \frac{2Mt}{2\sin(t/2)}\,dt \leqq \frac{M}{\pi}\int_0^{\pi/n}\frac{tdt}{(2/\pi)t/2} = \frac{M\pi}{n}.$$

In order to estimate Q_n, we introduce the function

$$\Lambda_n(t) = \frac{1}{\pi n}\int_t^\pi \frac{\sin nu}{(2\sin(u/2))^2}\,du,$$

and integrate by parts. By the second mean value theorem,

$$|\Lambda_n(t)| \leqq \frac{2}{\pi n^2}\cdot\frac{1}{(2\sin(t/2))^2} < \frac{\pi}{2n^2t^2}.$$

The function g is absolutely continuous and $|g'(x)| \leqq M$ almost everywhere. Thus

$$|Q_n| \leqq \frac{1}{\pi}\left|\Big[(g(x+t) - g(x-t))\Lambda_n(t)\Big]_{\pi/n}^{\pi}\right| + \frac{1}{\pi}\left|\int_{\pi/n}^{\pi}[g'(x+t) + g'(x-t)]\Lambda_n(t)dt\right|$$
$$\leqq \frac{1}{\pi}\cdot M\cdot\frac{2\pi}{n}\cdot\frac{\pi}{2n^2(\pi/n)^2} + \frac{2M}{\pi}\int_{\pi/n}^{\pi}|\Lambda_n(t)|\,dt$$
$$< \frac{M}{\pi n} + \frac{M}{n^2}\int_{\pi/n}^{\infty}\frac{dt}{t^2} = \frac{2}{\pi}\,\frac{M}{n}.$$

This completes the proof of the lemma, with $B = \pi + 2/\pi$.

Suppose now that the Fourier series of f is of power series type so that $\tilde{f} = -if$. If $|f(x+h)-f(x)| \leqq M|h|$, then

$$|\sigma_{n-1}(x) - f(x)| = |\tilde{\sigma}_{n-1}(x) - \tilde{f}(x)| \leqq BM/n. \tag{1.6}$$

To complete the proof of Theorem 1, we introduce the function

$$f_h(x) = \frac{1}{2h}\int_{-h}^{h} f(x+t)dt = [F(x+h) - F(x-h)]/2h$$
$$\sim \sum_{\nu=0}^{\infty} c_\nu e^{i\nu x}\left(\frac{\sin \nu h}{\nu h}\right),$$

where $F(x)$ is a primitive of f. Hence df_h/dx exists, is continuous, and does not exceed $\omega(2h)/2h \leqq \omega(h)/h$ in absolute value. Moreover, the Fourier series of f_h is also of power series type. Now,

$$\begin{aligned} &|\sigma_{n-1}(x; f) - f(x)| \\ &\quad\leqq |\sigma_{n-1}(x; f) - \sigma_{n-1}(x; f_h)| + |\sigma_{n-1}(x; f_h) - f_h(x)| + |f_h(x) - f(x)| \\ &\quad= \alpha_n + \beta_n + \gamma_n, \end{aligned}$$

say, and

$$\gamma_n = \left|\frac{1}{2h}\int_{-h}^{h}[f(x+t) - f(x)]dt\right| \leqq \omega(h),$$

$$\beta_n \leqq B\frac{\omega(h)}{h}\cdot\frac{1}{n} \qquad \text{(by (1.6))},$$

$$\alpha_n = |\sigma_{n-1}(x; f - f_h)| \leqq \max_x |f - f_h| \leqq \omega(h).$$

If we set $h = 2\pi/n$ and collect the results, we obtain (1.3) with $A = 2 + B/2\pi < 4$.

2. **Additional remarks.** The foregoing proof of the lemma has certain disadvantages. First of all, it uses the result that a Lipschitz function is an indefinite integral, a fact which lies deeper than the assumptions of the lemma. Moreover, the argument does not work with the L^p metric. These difficulties are avoided by the following somewhat longer variant of the proof of the lemma. For the sake of brevity we do not compute the absolute constants C that occur in the proof.

Let P_n and Q_n have the same meaning as before, and let $\psi(x, t) = f(x+t) - f(x-t)$. Hence

$$|P_n| \leqq \left|\frac{1}{\pi}\int_0^{\pi/n} \psi(x, t)\frac{\sin nt}{n(2\sin(t/2))^2}dt\right| \leqq \int_0^{\pi/n} |\psi(x, t)|\, t^{-1}dt.$$

Let $R_n(t) = 1/\pi n(2\sin(t/2))^2 < 1/nt^2$. Then, for $n \geqq 1$,

$$Q_n = \int_{\pi/n}^{\pi} \psi(x, t)R_n(t) \sin ntdt$$

$$= -\int_0^{\pi(n-1)/n} \psi(x, t + \pi/n)R_n(t + \pi/n) \sin ntdt,$$

$$2Q_n = \int_{\pi/n}^{\pi(n-1)/n} \psi(x, t)[R_n(t) - R_n(t + \pi/n)] \sin ntdt$$

$$+ \int_{\pi/n}^{\pi(n-1)/n} [\psi(x, t) - \psi(x, t + \pi/n)]R_n(t + \pi/n) \sin ntdt$$

$$- \int_0^{\pi/n} \psi(x, t + \pi/n)R_n(t + \pi/n) \sin ntdt$$

$$+ \int_{\pi(n-1)/n}^{\pi} \psi(x, t)R_n(t) \sin ntdt = I_n + J_n + K_n + L_n,$$

say.

By the mean-value theorem

$$| R_n(t) - R_n(t + \pi/n) | \leqq Cn^{-2}t^{-3},$$

so that

$$| I_n | \leqq Cn^{-2} \int_{\pi/n}^{\pi-\pi/n} | \psi(x, t) | t^{-3}dt \leqq Cn^{-2} \int_{\pi/n}^{\pi} | \psi(x, t) | t^{-3}dt.$$

Since $R_n(t+\pi/n) \leqq 1/nt^2$, and

$$\psi(x, t) - \psi(x, t + \pi/n) = \psi(x + t - \pi/2n, \pi/2n) - \psi(x - t - \pi/2n, \pi/2n),$$

we find

$$| J_n | \leqq Cn^{-1} \int_{\pi/n}^{\pi} | \psi(x + t - \pi/2n, \pi/2n) | t^{-2}dt$$

$$+ Cn^{-1} \int_{\pi/n}^{\pi} | \psi(x - t - \pi/2n, \pi/2n) | t^{-2}dt.$$

Moreover, since $R_n(t+\pi/n) < Cn$ for $0 \leqq t \leqq \pi/n$,

$$| K_n | \leqq Cn \int_0^{\pi/n} | \psi(x, t + \pi/n) | dt.$$

Finally,

$$| L_n | \leqq Cn^{-1} \int_{\pi(n-1)/n}^{\pi} | \psi(x, t) | dt = Cn^{-1} \int_0^{\pi/n} | \psi(x + \pi, t) | dt.$$

By assumption, $|\psi(x, u)| \leqq M|u|$, uniformly in x. From this we immediately deduce that each of the terms $|P_n|$, $|I_n|$, $|J_n|$, $|K_n|$, $|L_n|$ is less than or equal to CM/n, and (1.5) is proved.

Suppose now that instead of the inequality $|g(x+h)-g(x)| \leqq M|h|$ we have

$$(2.1)\quad M_p[g(x+h)-g(x)] = \left\{\int_0^{2\pi} |g(x+h)-g(x)|^p dx\right\}^{1/p} \leqq M|h|$$

for some $p \geqq 1$. Then Minkowski's inequality for integrals shows that $M_p[P_n]$, $M_p[I_n]$, $M_p[J_n]$, $M_p[K_n]$, $M_p[L_n]$ are all less than or equal to CM/n. For example,

$$M_p[P_n] \leqq \int_0^{\pi/n} M_p[\psi(x, t)]t^{-1}dt \leqq \int_0^{\pi/n} 2Mdt = 2M\pi/n,$$

$$M_p[I_n] \leqq Cn^{-2}\int_{\pi/n}^{\pi} M_p[\psi(x, t)]t^{-3}dt$$

$$\leqq 2CMn^{-2}\int_{\pi/n}^{\infty} t^{-2}dt = CM/n,$$

and similarly in other cases. Thus, *under the hypothesis* (2.1),

$$M_p[\tilde{\sigma}_{n-1}(x) - \tilde{g}(x)] \leqq BM/n$$

where B is an absolute constant. By an argument similar to that by which Theorem 1 was deduced from the lemma, we obtain the following theorem.

THEOREM 2. *Suppose that the Fourier series of $f(x)$ is of the power series type. Then*

$$M_p[\sigma_{n-1}(x) - f(x)] \leqq A\omega_p(2\pi/n) \qquad (p \geqq 1)$$

where $\omega_p(\delta) = \sup_{|t|\leqq\delta} M_p[f(x+t)-f(x)]$.

Theorems 1 and 2 hold for (C, α) means, whatever $\alpha > 0$. The analogues for Abel means are immediate consequences of the Cauchy-Riemann equations.

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