

A CORRECTION TO MY BOOK ON TOPOLOGICAL GROUPS

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On p. 28 of my monograph on topological groups,² implicit use is made of the following proposition: "If G is a topological group, and g a closed invariant subgroup of G , then the factor-group G/g cannot have a dimension greater than that of G ." This is a very plausible but hitherto unproved result; the assertions on the dimension of a projective limit (IGTA, p. 28) must therefore partake of its conjectural nature. However, the results on the dimension of compact groups and of locally compact abelian groups (IGTA, chap. 5, §25, and chap. 6, §29) can be justified by dealing with a special case of that proposition; to do so is the purpose of the present note. Since no satisfactory theory of dimension exists except for "separable metrizable" spaces,³ it will be enough to consider the case of groups which are metrizable and separable. A compact group which satisfies these conditions will be called *enumerably compact* in this note; such a group has at most enumerably many irreducible linear representations; it follows from this, and from the results of IGTA, §25, that it is the projective limit of a *sequence* of compact Lie groups. The result to be proved is now as follows.

THEOREM. *Let G be an enumerably compact group. Let g_0 be a closed invariant subgroup of G such that G/g_0 is a Lie group. Let U_0 be a neighborhood of the unit-element in G/g_0 , homeomorphic to the interior of a sphere; and let X be the set of all elements of G , the image of which in G/g_0 is in U_0 . Then X is homeomorphic to the (topological) direct product $U_0 \times g_0$.*

Let ϕ be the canonical mapping of G on the factor-group $G_0 = G/g_0$, i.e., the mapping in which, to every element of G , corresponds its co-set with respect to g_0 . As is well known, our theorem will be proved if we show that we can find in G a system of representatives U for U_0 , i.e., a homeomorphic image $U = \psi(U_0)$ of U_0 in G such that the inverse ψ^{-1} of ψ coincides on U with ϕ . In order to do that, we write G

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² *L'intégration dans les groupes topologiques et ses applications*, Actualités Scientifiques et Industrielles, Hermann, Paris, 1940; this book will be quoted as IGTA. I am much indebted to H. Samelson for having first pointed out to me the conjectural nature of the result in question, and also for the following correction to the same book: on p. 26, lines 8 and 15, the word "invariant" should be omitted.

³ Cf. W. Hurewicz and H. Wallman, *Dimension theory*, Princeton, 1941 (see the Appendix).

as projective limit of a sequence of Lie groups $G_n = G/g_n$, where the g_n , for $n = 0, 1, 2, \dots$, are a sequence of closed invariant subgroups of G , beginning with the given subgroup g_0 , such that $g_n \supset g_{n+1}$ for every n , and with no common element other than the unit-element of G . If we put $\gamma_n = g_{n+1}/g_n$, we can consider γ_n as a subgroup of $G_n = G/g_n$; it is a closed invariant subgroup of G_n , and we have (by IGTA, §3) $G_n = G_{n+1}/\gamma_{n+1}$; let ϕ_n be the canonical mapping of G_{n+1} on the factor-group $G_n = G_{n+1}/\gamma_{n+1}$. We now define by induction, for every n , a homeomorphic image U_n of U_0 in G_n as follows. Assuming that U_n has been defined, we may consider its inverse image by ϕ_n , in G_{n+1} , as a fibre-space with the base-space U_n and with fibres which are homeomorphic to γ_{n+1} ; as U_n is homeomorphic to U_0 , that is, to the interior of a sphere of a certain dimension, it follows from this, by Feldbau's theorem,⁴ that this inverse image is homeomorphic to the topological product of U_n and of γ_{n+1} , and that we can find in it a continuous system of representatives, which we call U_{n+1} , for the base U_n . Then U_{n+1} is a homeomorphic image $U_{n+1} = \psi_n(U_n)$ of U_n in G_{n+1} , and the inverse ψ_n^{-1} of the homeomorphic mapping ψ_n of U_n on U_{n+1} is no other than ϕ_n . That being so, let x_0 be any point in U_0 , and put $x_{n+1} = \psi_n(x_n)$ for $n = 0, 1, 2, \dots$; as we have $x_n = \phi_n(x_{n+1})$ for every n , the sequence (x_0, x_1, x_2, \dots) defines a point x in the projective limit G of the sequence G_0, G_1, G_2, \dots ; if now we write $x = \psi(x_0)$, ψ is a mapping of U_0 in G , with the properties which were required for the proof of our theorem.

COROLLARY. *If a factor-group of an enumerably compact group G is a Lie group of dimension n , then G itself has at least the dimension n .*

The above proof suggests that the conjectural proposition mentioned at the beginning of this note is related to the following problem. Let G be a topological group, g a closed subgroup of G , and $H = G/g$ the homogeneous space determined by G and g (IGTA, §2), that is, the space of cosets of g in G ; can G be considered as a fibre-space, with base H and with fibres homeomorphic to g , according to any one of the known definitions of fibre-spaces?⁵ One may expect the answer to be in the affirmative, provided a point in H has a system of neighborhoods which are all homotopic to zero (that is, contractible to a point) in H .

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⁴ J. Feldbau, *Sur la classification des espaces fibrés*, C. R. Acad. Sci. Paris vol. 208 (1939) p. 1621.

⁵ Cf., for example, R. H. Fox, *On fibre-spaces*, Bull. Amer. Math. Soc. vol. 49 (1943) p. 555.