CONCERNING THE DEFINITION OF HARMONIC FUNCTIONS

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1. Introduction. A real function $u(x, y)$, defined in a domain (non-null connected open set) D, is said to be harmonic in D provided $u(x, y)$ and its partial derivatives of the first and second orders are continuous and the Laplace equation,

$$\Delta u \equiv \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0, \tag{1}$$

is satisfied throughout D. A function is said to be harmonic at a point provided it is harmonic in a domain containing the point.

It has been shown [1][1] that if $u(x, y)$ is continuous in D and if the second order partial derivatives $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial y^2$ exist and satisfy the Laplace equation (1) throughout D, then $u(x, y)$ is harmonic in D.

We shall show that if $u(x, y)$ and its partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$ are continuous in D, if $\partial u/\partial x$ and $\partial u/\partial y$ are differentiable, or even have finite Dini derivates, with respect to x and y at all points of D except at most at the points of a denumerable set of points in D, and if the Laplace equation (1) is satisfied at almost all points of D at which $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial y^2$ exist, then $u(x, y)$ is harmonic in D.

Our result is comparable with the Looman-Menchoff theorem [**3**, pp. 9–16; **5**, pp. 198–201] concerning the Cauchy-Riemann first order partial differential equations and analytic functions of a complex variable. Ridder [**4**] has stated that harmonic functions can be given a Looman-Menchoff characterization; but a generalization of the Looman-Menchoff theorem on which his proof is based is invalid, for there are functions having isolated singularities which satisfy the hypotheses of the generalization without satisfying the conclusion. For a generalization of the Looman-Menchoff theorem, see Maker [**2**].

2. Notation and lemmas. By $C(Q)$ we shall denote a square, by $C(R)$ a rectangle, having sides parallel to the coordinate axes. The set consisting of the points of $C(Q)$, or of $C(R)$, plus its interior, will be denoted by Q, or R, respectively.

Let F be a non-null set closed with respect to the domain D, and $C(Q)$ any square with Q lying in D, with sides of positive length and parallel to the coordinate axes, and with center at a point of F. Then the points common to F and Q will be called a *portion* of F.

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[1] Numbers in brackets refer to the references cited at the end of the paper.

We shall use as lemmas the following known results.

LEMMA 1. *If $u(x, y)$ is continuous at (x_0, y_0) and harmonic in a deleted neighborhood of (x_0, y_0), then $u(x, y)$ is harmonic at (x_0, y_0).*

PROOF. This follows from the fact that the function can be expanded in a two-way power series in a deleted neighborhood of (x_0, y_0).

LEMMA 2. *If $u(x, y)$ is harmonic in the finite domain D, then for each rectangle $C(R)$ such that R lies in D we have*

$$\int_{C(R)} \frac{du}{d\nu} ds = 0, \tag{2}$$

where $d/d\nu$ denotes differentiation in the direction of the outward normal.

PROOF. The result follows directly from Green's theorem,

$$\int_{C(R)} \frac{du}{d\nu} ds = \int\int_R \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) dxdy.$$

LEMMA 3. *If $u(x, y)$ and its partial derivatives of the first order are continuous in D and if for every square $C(Q)$ having sides parallel to the coordinate axes and such that Q lies in D we have*

$$\int_{C(Q)} \frac{du}{d\nu} ds = 0, \tag{3}$$

then $u(x, y)$ is harmonic in D.

PROOF. The mean-value function,

$$u^{(r)}(x, y) \equiv \frac{1}{4r^2} \int_{-r}^{r} \int_{-r}^{r} u(x + \xi, y + \eta) d\xi d\eta,$$

has continuous partial derivatives of the second order in the part $D^{(r)}$ of D in which $u^{(r)}(x, y)$ is defined. We have

$$\frac{\partial^2 u^{(r)}(x, y)}{\partial x^2} = \frac{1}{4r^2} \int_{-r}^{r} \left[\frac{\partial u(x + r, y + \eta)}{\partial x} - \frac{\partial u(x - r, y + \eta)}{\partial x}\right] d\eta$$

and

$$\frac{\partial^2 u^{(r)}(x, y)}{\partial y^2} = \frac{1}{4r^2} \int_{-r}^{r} \left[\frac{\partial u(x + \xi, y + r)}{\partial y} - \frac{\partial u(x + \xi, y - r)}{\partial y}\right] d\xi,$$

whence

$$\Delta u^{(r)}(x, y) = \frac{1}{4r^2} \int_{C(Q_r)} \frac{du}{dv} ds, \tag{4}$$

where $C(Q_r)$ is the square with sides of length $2r$ and parallel to the coordinate axes, and with center at (x, y).

From (3) and (4) it follows that $u^{(r)}(x, y)$ is harmonic in $D^{(r)}$. Hence $u(x, y)$, the uniform limit of $u^{(r)}(x, y)$ on any closed and bounded subset of D as $r \to 0$, is harmonic in D.

LEMMA 4 (BAIRE'S THEOREM). *Let F be a non-null plane set lying in a domain D and closed with respect to D, and let $\{F_n\}$, $n = 1, 2, \cdots$, be a sequence of sets lying in D and closed with respect to D such that $\{F_n\}$ covers F; that is, each point of F is a point of at least one F_n. Then there is a member F_N of $\{F_n\}$ which contains a portion of F.*

PROOF. If the result were not valid, there would be a descending sequence $\{P_n\}$, $P_1 \supset P_2 \supset \cdots$, of portions of F such that P_n and F_n have no point in common, $n = 1, 2, \cdots$. Then $\prod_{n=1}^{\infty} P_n$ and $\sum_{n=1}^{\infty} F_n$ would have no point in common. But this is impossible since $\prod_{n=1}^{\infty} P_n$ contains a point of F and $\sum_{n=1}^{\infty} F_n$ covers F.

LEMMA 5. *Let $C(Q)$ be a square having sides parallel to the coordinate axes, let F be a closed non-null set in Q, let $C(R)$ be the smallest rectangle (which may be degenerate) having sides parallel to the coordinate axes and satisfying the condition that F is contained in R, and let the vertices of $C(R)$ have coordinates*

$$(x_1, y_1), \ (x_2, y_1), \ (x_2, y_2), \ (x_1, y_2), \qquad x_1 \leqq x_2, \ y_1 \leqq y_2.$$

If the real function $w(x, y)$ is defined on the set Q, if the first order partial derivatives of $w(x, y)$ exist, or even if the Dini derivates are finite, at every point of Q except at most at the points of a denumerable set of points in Q, and if for the finite constant N we have

$$| w(x_0 + h, y_0) - w(x_0, y_0) | \leqq N | h |,$$
$$| w(x_0, y_0 + k) - w(x_0, y_0) | \leqq N | k |,$$

for all (x_0, y_0) in F and all $(x_0 + h, y_0)$, $(x_0, y_0 + k)$ in Q, then

$$\left| \int_{x_1}^{x_2} [w(x, y_2) - w(x, y_1)] dx - \iint_F \frac{\partial w}{\partial y} dxdy \right| \leqq 5N \operatorname{meas} (Q - F),$$

$$\left| \int_{y_1}^{y_2} [w(x_2, y) - w(x_1, y)] dy - \iint_F \frac{\partial w}{\partial x} dxdy \right| \leqq 5N \operatorname{meas} (Q - F).$$

PROOF. A proof of Lemma 5 may be found in Saks [5, pp. 198–199] or Menchoff [3, pp. 10–12].

We note, relative to Lemma 5, that since $w(x, y)$ is differentiable, or has finite Dini derivates, with respect to x and y at all points of Q except at most at the points of a denumerable set of points in Q, it follows [5, pp. 236, 272] that $\partial w/\partial x$ and $\partial w/\partial y$ exist almost everywhere in Q and are integrable.

3. **Theorem.** We shall establish the following result.

THEOREM. *If the real function $u(x, y)$ and its first order partial derivatives with respect to x and y are continuous in the finite domain D, and if $\partial u/\partial x$ and $\partial u/\partial y$ are differentiable, or even have finite Dini derivatives, with respect to x and y at all points of D except at most at the points of a denumerable set of points in D, and if the Laplace equation,*

$$\Delta u \equiv \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0,$$

is satisfied at almost all points at which $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial y^2$ exist in D, then $u(x, y)$ is harmonic in D.

PROOF. Suppose that there is a point of D at which $u(x, y)$ is not harmonic; we shall obtain a contradiction.

Denote the set of points of D at which $u(x, y)$ is not harmonic by F. Since by definition the set of points at which $u(x, y)$ is harmonic is open, it follows that F is closed with respect to D. Further, by Lemma 1, F has no isolated points. Hence F is perfect with respect to D.

For each positive integer n, let F_n be the set of points (x, y) of F for which

$$\left| \frac{\partial u(x+h, y)}{\partial x} - \frac{\partial u(x, y)}{\partial x} \right| \leqq n \left| h \right|,$$

$$\left| \frac{\partial u(x, y+h)}{\partial x} - \frac{\partial u(x, y)}{\partial x} \right| \leqq n \left| h \right|,$$

$$\left| \frac{\partial u(x+h, y)}{\partial y} - \frac{\partial u(x, y)}{\partial y} \right| \leqq n \left| h \right|,$$

$$\left| \frac{\partial u(x, y+h)}{\partial y} - \frac{\partial u(x, y)}{\partial y} \right| \leqq n \left| h \right|,$$

for all h satisfying $|h| \leqq 1/n$. Since $\partial u/\partial x$ and $\partial u/\partial y$ are continuous in D, it follows that the sets F_n are closed with respect to D; and since $\partial u/\partial x$ and $\partial u/\partial y$ are differentiable, or have finite Dini derivates, with respect to x and y at all points of D except at most at the points of a denumerable set of points in D, it follows that the sets $\{F_n\}$,

$n=1, 2, \cdots$, cover all of F except the points of a set H which is at most denumerable.

Then we have

$$F = \sum_{n=1}^{\infty} F_n + H. \tag{5}$$

It follows from (5) and Lemma 4 that thee is a portion P of F either consisting of a single isolated point of H, or contained in an F_N of $\{F_n\}$. But since F is perfect with respect to D, the former alternative is impossible, so that the latter alternative holds.

The above portion P of F is contained in F_N, and is the common part of F and a set Q_0 in D, where $C(Q_0)$ is a square in D with center at a point of F and with sides parallel to the coordinate axes.

Let $C(Q)$ be any square lying in Q_0 and having its sides parallel to the coordinate axes, and let $F\cdot Q$ be the common part of F and Q. Let the sides of $C(Q)$ be divided into n equal parts, with n so large that the length of each part is less than or equal to $1/N$. Lines through the points of division parallel to the coordinate axes divide Q into n^2 squares. Let $Q_{p,n}$, $p=1, 2, \cdots, l; l \leqq n^2$, denote those of the n^2 squares having points in common with F.

For each $Q_{p,n}$ let $C(R_{p,n})$ be the smallest rectangle (which may be degenerate) having sides parallel to the coordinate axes and such that $R_{p,n}$ contains $F\cdot Q_{p,n}$. Let the vertices of $C(R_{p,n})$ have coordinates

$$(x_{1,p,n}, y_{1,p,n}), \quad (x_{2,p,n}, y_{1,p,n}), \quad (x_{2,p,n}, y_{2,p,n}), \quad (x_{1,p,n}, y_{2,p,n}),$$
$$x_{1,p,n} \leqq x_{2,p,n}, \ y_{1,p,n} \leqq y_{2,p,n}.$$

By Lemma 2 and the uniform continuity of $u(x, y)$ in Q, we have

$$\int_{C(Q)} \frac{du}{d\nu} ds = \sum_{p=1}^{l} \int_{C(R_{p,n})} \frac{du}{d\nu} ds. \tag{6}$$

Since

$$\int_{C(R_{p,n})} \frac{du}{d\nu} ds = \int_{x_{1,p,n}}^{x_{2,p,n}} \left[\frac{\partial u(x, y_{2,p,n})}{\partial y} - \frac{\partial u(x, y_{1,p,n})}{\partial y}\right] dx + \int_{y_{1,p,n}}^{y_{2,p,n}} \left[\frac{\partial u(x_{2,p,n}, y)}{\partial x} - \frac{\partial u(x_{1,p,n}, y)}{\partial x}\right] dy,$$

by Lemma 5 we have

$$\left| \int_{C(R_{p,n})} \frac{du}{d\nu} ds - \iint_{F\cdot Q_{p,n}} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) dxdy \right| \leqq 10N \operatorname{meas} (Q_{p,n} - F\cdot Q_{p,n}). \tag{7}$$

From (6) and (7) we obtain

$$(8)\qquad \left| \int_{C(Q)} \frac{du}{d\nu}\, ds - \int\!\!\int_{F\cdot Q} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) dxdy \right| \leqq 10N \text{ meas} \left(\sum_{p=1}^{l} Q_{p,n} - F\cdot Q \right).$$

Since the lengths of the sides of the squares $C(Q_{p,n}) \to 0$ as $n \to \infty$, it follows that

$$(9)\qquad \lim_{n\to\infty} \text{meas} \left(\sum_{p=1}^{l} Q_{p,n} - F\cdot Q \right) = 0.$$

By hypothesis and the note at the end of §2, (1) is satisfied almost everywhere in D. Consequently from (8) and (9) we obtain

$$\int_{C(Q)} \frac{du}{d\nu}\, ds = 0.$$

Hence by Lemma 3, $u(x, y)$ is harmonic in Q_0. But the center of Q_0 is a point of F, so that $u(x, y)$ is not harmonic in Q_0. Thus the supposition that there is a point of D at which $u(x, y)$ is not harmonic has led to a contradiction.

References

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