THE PSEUDO-ANGLE IN SPACE OF $2n$ DIMENSIONS

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1. Introduction. The theory of functions of a single complex variable is essentially identical with the conformal geometry of the real (or complex) plane. However, this is not the case in the theory of functions of *two or more* complex variables. Any set of $n \geqq 2$ functions of n complex variables with nonvanishing jacobian induces a correspondence between the points of a real (or complex) $2n$-dimensional euclidean space R_{2n}. The infinite group G of all such correspondences is obviously *not* the conformal group of R_{2n}, which is merely the inversive group of $(n+1)(2n+1)$ parameters.[1] Poincaré in his fundamental paper in Palermo Rendiconti (1907) has called G the group of *regular* transformations. However, in an abstract presented before the American Mathematical Society, 1908, Kasner found it more appropriate to term it the *pseudo-conformal group* G. This name is now standard.

In his work of 1908, which he later published in full in 1940, Kasner investigated the possibility of characterizing the pseudo-conformal group G of four dimensions (the case $n=2$ complex variables) in a purely geometric way.[2] His principal result is as follows: *A transformation of R_4 is pseudo-conformal if and only if it preserves the pseudo-angle between any curve and a three-dimensional hypersurface at their point of intersection.* This theorem demonstrates how the pseudo-angle may be used to characterize G within the group of arbitrary point transformations of R_4.

We shall show in this paper how Kasner's pseudo-angle theorem can be carried over to $2n$ dimensions almost without any change. The pseudo-angle is important also because all other differential invariants of the first order under the pseudo-conformal group are really combinations of this pseudo-angle.[3]

2. The minimal coordinates. Let $(x_1, x_2, \cdots, x_n; y_1, y_2, \cdots, y_n)$

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[1] The conformal group of a euclidean space R_m of any dimension $m>2$, odd or even, is the inversive group of $(m+1)(m+2)/2$ parameters (Liouville's theorem). Fialkow has studied the conformal geometry of any curve not only in a euclidean space R_m but also in any riemann space V_m. See his paper, *Conformal geometry of curves*, Trans. Amer. Math. Soc. vol. 51 (1942).

[2] Kasner, *Conformality in connection with functions of two complex variables*, Trans. Amer. Math. Soc. vol. 48 (1940) pp. 50–62.

[3] See Kasner and DeCicco, *Pseudo-conformal geometry: Functions of two complex variables*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 317–328.

$=(x_\alpha, y_\alpha)$ denote the cartesian coordinates of a real or complex euclidean $2n$-dimensional space R_{2n}. We shall find it convenient to introduce the minimal coordinates $(u_1, u_2, \cdots, u_n; v_1, v_2, \cdots, v_n)=(u_\alpha, v_\alpha)$ defined by

$$(1)\qquad u_\alpha = x_\alpha + iy_\alpha, \qquad v_\alpha = x_\alpha - iy_\alpha,$$

for $\alpha=1, 2, \cdots, n$. The inverse of this correspondence is

$$(2)\qquad x_\alpha = (u_\alpha + v_\alpha)/2, \qquad y_\alpha = (u_\alpha - v_\alpha)/2i.$$

The following relations are noted between the partial derivatives in minimal coordinates and cartesian coordinates:

$$(3)\qquad \frac{\partial}{\partial u_\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x_\alpha} - i\frac{\partial}{\partial y_\alpha}\right), \qquad \frac{\partial}{\partial v_\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x_\alpha} + i\frac{\partial}{\partial y_\alpha}\right).$$

The operators $\partial/\partial u_\alpha$ may be called the *mean derivatives*; and the operators $\partial/\partial v_\alpha$ may be termed the *phase derivatives*.[4]

In minimal coordinates, the square of the linear element ds is

$$(4)\qquad ds^2 = \sum_{\alpha=1}^{n} du_\alpha dv_\alpha.$$

The angle θ between any two curve elements through a common point is

$$(5)\qquad \cos\theta = \frac{\sum_{\alpha=1}^{n}\left[du_\alpha^{(1)} dv_\alpha^{(2)} + du_\alpha^{(2)} dv_\alpha^{(1)}\right]}{2\left[\left(\sum_{\alpha=1}^{n} du_\alpha^{(1)} dv_\alpha^{(1)}\right)\left(\sum_{\alpha=1}^{n} du_\alpha^{(2)} dv_\alpha^{(2)}\right)\right]^{1/2}}.$$

3. **The pseudo-conformal group.** This is given in minimal coordinates by

$$(6)\qquad U_\alpha = U_\alpha(u_1, u_2, \cdots, u_n), \qquad V_\alpha = V_\alpha(v_1, v_2, \cdots, v_n),$$

for $\alpha=1, 2, \cdots, n$, where the jacobians $|\partial U_\alpha/\partial u_\beta|$ and $|\partial V_\alpha/\partial v_\beta|$ are each not zero. Our problem is to inaugurate the study of this group in detail.

In what follows, we shall omit from consideration the special minimal n-flats $u_\alpha=$const. and $v_\alpha=$const. Our pseudo-conformal group may be defined as the direct part of the total mixed group preserving these $2\infty^n$ special minimal n-flats.

[4] Kasner, *The second derivative of a polygenic function*, Trans. Amer. Math. Soc. vol. 30 (1928). Also Kasner and DeCicco, *The derivative circular congruence-representation of a polygenic function*, Amer. J. Math. vol. 61 (1939) pp. 995–1003.

4. The pseudo-conformal geometry of differential elements of first order. We shall be interested mainly in the geometry of the curve elements at a fixed point of R_{2n}. These form a $(2n-1)$-dimensional manifold $\Sigma_{(2n-1)}$.

Let $(\phi_1, \phi_2, \cdots, \phi_n; \psi_1, \psi_2, \cdots, \psi_n) = (\phi_\alpha, \psi_\alpha)$ be any set of numbers (not all zero) proportional to the differentials $(du_1, du_2, \cdots, du_n; dv_1, dv_2, \cdots, dv_n) = (du_\alpha, dv_\alpha)$ respectively so that $\phi_\alpha = \rho du_\alpha$, $\psi_\alpha = \rho dv_\alpha$. Then any such set and only those sets proportional to it define, in homogeneous coordinates, any curve element e of $\Sigma_{(2n-1)}$.

The pseudo-conformal group G induces the $(2n^2-1)$-parameter group $G_{(2n^2-1)}$ among the curve elements of $\Sigma_{(2n-1)}$ defined as follows:

$$\rho\Phi_\alpha = \sum_{\beta=1}^{n} a_{\alpha\beta}\phi_\beta, \qquad \rho\Psi_\alpha = \sum_{\beta=1}^{n} b_{\alpha\beta}\Psi_\beta, \tag{7}$$

for $\alpha = 1, 2, \cdots, n$, where the determinants $|a_{\alpha\beta}|$ and $|b_{\alpha\beta}|$ are each not zero.

5. The isoclines. An isocline γ_{2r} of $2r$ dimensions where $0 < r < n$ is defined by the system of $(2n-2r)$ linear equations of the special forms

$$\sum_{\beta=1}^{n} \lambda_{\alpha\beta}\phi_\beta = 0, \qquad \sum_{\beta=1}^{n} \mu_{\alpha\beta}\psi_\beta = 0, \tag{8}$$

for $\alpha = 1, 2, \cdots, n-r$, where each of the matrices $(\lambda_{\alpha\beta})$ and $(\mu_{\alpha\beta})$ is of rank $(n-r)$.

A consideration of these equations will show that there are $\infty^{2r(n-r)}$ isoclines γ_{2r} in $\Sigma_{(2n-1)}$. Also r curve elements which do not lie in a lower dimensional isocline determine a unique $2r$-dimensional isocline. Of course, two distinct isoclines will intersect in an isocline whose greatest possible dimension is equal to the lowest of the two given isoclines or else they will have no common curve elements.

THEOREM 1. *Under the induced pseudo-conformal group $G_{(2n^2-1)}$, any two $2r$-dimensional isoclines are equivalent. Any isocline γ_{2r} is a pseudo-conformal manifold $\Sigma_{(2r-1)}$ contained in the larger pseudo-conformal manifold $\Sigma_{(2n-1)}$.*

The proof of Theorem 1 is as follows. By applying (7) to (8), any $2r$-dimensional isocline becomes a $2r$-dimensional isocline under $G_{(2n^2-1)}$. Any isocline γ_{2r} may be carried into the canonical isocline $\gamma_{2r}(0)$

$$\phi_{r+1} = \phi_{r+2} = \cdots = \phi_n = 0, \quad \psi_{r+1} = \psi_{r+2} = \cdots = \psi_n = 0. \tag{9}$$

By the preceding, we may note that $(\phi_1, \phi_2, \cdots, \phi_r; \psi_1, \psi_2, \cdots, \psi_r)$ may be used as homogeneous coordinates of any curve element of the canonical isocline $\gamma_{2r}(0)$. By this remark and by determining the subgroup of $G_{(2n^2-1)}$ preserving $\gamma_{2r}(0)$, it is seen that the proof of our Theorem 1 is complete.

6. **The pseudo-conformal geometry of two curve elements.** In the first place, it is observed that the most general transformation of $G_{(2n^2-1)}$ which will carry the curve element $(1, 0, \cdots, 0; 1, 0, \cdots, 0)$ into the curve element $(\rho_1\Phi_\alpha^{(1)}, \rho_1\Psi_\alpha^{(1)})$ is of the form

$$\rho\Phi_\alpha = \rho_1\Phi_\alpha^{(1)}\phi_1 + \sum_{\beta=2}^{n} a_{\alpha\beta}\phi_\beta, \qquad \rho\Psi_\alpha = \rho_1\Psi_\alpha^{(1)}\psi_1 + \sum_{\beta=2}^{n} b_{\alpha\beta}\Psi_\beta. \tag{10}$$

Any curve element which lies in the isocline of dimension two determined by $(1, 0, \cdots, 0; 1, 0, \cdots, 0)$ must be of the form $(\phi_1^{(2)}, 0, \cdots, 0; \psi_1^{(2)}, 0, \cdots, 0)$. The transform of this is

$$\rho\Phi_\alpha^{(2)} = \rho_1\Phi_\alpha^{(1)}\phi_1^{(2)}, \qquad \rho\Psi_\alpha^{(2)} = \rho_1\Psi_\alpha^{(1)}\psi_1^{(2)}. \tag{11}$$

These immediately guarantee that the transformed element $(\Phi_\alpha^{(2)}, \Psi_\alpha^{(2)})$ is in the isocline determined by $(\Phi_\alpha^{(1)}, \Psi_\alpha^{(1)})$. Moreover the expression

$$(\Phi_\alpha^{(2)}/\Psi_\alpha^{(2)})\cdot(\Psi_\alpha^{(1)}/\Phi_\alpha^{(1)}), \tag{12}$$

which is the same for $\alpha=1, 2, \cdots, n$, is invariant. By taking the logarithm of this invariant, and then multiplying the result by $1/2i$, it is found by (5) that the resulting invariant represents the angle between the two given curve elements.

THEOREM 2. *Two curves elements e_1 and e_2 which lie in the same two-dimensional isocline γ_2 possess the unique invariant*

$$\theta = \frac{1}{2i}\log\frac{\phi_\alpha^{(2)}}{\psi_\alpha^{(2)}}\cdot\frac{\psi_\alpha^{(1)}}{\phi_\alpha^{(1)}}, \tag{13}$$

which is the same for all $\alpha=1, 2, \cdots, n$. It actually is the angle between the two curve elements e_1 and e_2.

It may be proved that any pair of curve elements not in the same two-dimensional isocline may be carried into any other such pair. Thus two curve elements will possess a differential invariant of the first order if and only if they lie in the same two-dimensional isocline γ_2. In that case, they have a unique invariant which is actually the angle between them.

7. **Kasner's pseudo-angle.** Any $(2n-1)$-dimensional hypersurface

element $S_{(2n-1)}$ of $\Sigma_{(2n-1)}$ may be given by the equation

$$\sum_{\alpha=1}^{n}(k_\alpha\phi_\alpha + l_\alpha\psi_\alpha) = 0. \tag{14}$$

The homogeneous coordinates of any hypersurface element of $(2n-1)$ dimensions are $(k_1, k_2, \cdots, k_n; l_1, l_2, \cdots, l_n) = (k_\alpha, l_\alpha)$.

Since we wish to omit from consideration those hypersurface elements which contain the special minimal n-flat elements, neither all the k's nor all the l's are zero.

The transformation formulas between the $(2n-1)$-dimensional hypersurface elements $S_{(2n-1)}$ in $\Sigma_{(2n-1)}$ are

$$\sigma k_\beta = \sum_{\alpha=1}^{n} a_{\alpha\beta}K_\alpha, \qquad \sigma l_\beta = \sum_{\alpha=1}^{n} b_{\alpha\beta}L_\alpha, \tag{15}$$

for $\beta = 1, 2, \cdots, n$.

Let $e(\phi_\alpha, \psi_\alpha)$ be any curve element of $\Sigma_{(2n-1)}$. The two-dimensional isocline determined by e is given parametrically by

$$\Phi_\alpha = r\phi_\alpha, \qquad \Psi_\alpha = s\psi_\alpha, \tag{16}$$

where r and s are the variable parameters. This isocline intersects the hypersurface element $S_{(2n-1)}$ given by equation (14) in the curve element $e^*(\Phi_\alpha, \Psi_\alpha)$ given by

$$\frac{\Phi_\beta}{\phi_\beta} = -\lambda\sum_{\alpha=1}^{n} l_\alpha\psi_\alpha, \qquad \frac{\Psi_\beta}{\psi_\beta} = \lambda\sum_{\alpha=1}^{n} k_\alpha\phi_\alpha. \tag{17}$$

By Theorem 2, the angle between the two curve elements e and e^* is invariant. Therefore it is an invariant between e and $S_{(2n-1)}$.

We shall now show that the angle obtained above is the unique invariant. Any curve element e may be carried into the canonical curve element $e^{(0)}(1, 0, \cdots, 0; 1, 0, \cdots, 0)$. By (10), it is seen that the group preserving this canonical curve element $e^{(0)}$ must satisfy the conditions $a_{11} = b_{11} = \rho_1$, $a_{\alpha 1} = b_{\alpha 1} = 0$, for $\alpha = 2, 3, \cdots, n$. Hence (15) may be written in the form

$$\begin{aligned} \sigma k_1 &= \rho_1 K_1, & \sigma l_1 &= \rho_1 L_1, \\ \sigma k_\beta &= \sum_{\alpha=1}^{n} a_{\alpha\beta}K_\alpha, & \sigma l_\beta &= \sum_{\alpha=1}^{n} b_{\alpha\beta}L_\alpha, \end{aligned} \tag{18}$$

for $\beta = 2, 3, \cdots, n$.

By choosing $a_{1\beta} = \sigma k_\beta / K_1$ for $\beta = 2, 3, \cdots, n$, the above transformation carries the hypersurface element (k_α, l_α) into the hypersurface element $(K_1, 0, \cdots, 0; L_1, 0, \cdots, 0)$. This clearly proves that a curve element e and a hypersurface element S_{2n-1} possess only one invariant.

THEOREM 3. *A curve element $e(\phi_\alpha, \psi_\alpha)$ and a $(2n-1)$-dimensional hypersurface element $S_{2n-1}(k_\alpha, l_\alpha)$ possess only the single invariant*

$$\theta = \frac{1}{2i} \log \left[- \sum_{\alpha=1}^{n} l_\alpha v_\alpha \Big/ \sum_{\alpha=1}^{n} k_\alpha u_\alpha \right]. \tag{19}$$

This is Kasner's pseudo-angle between e and S_{2n-1}. It represents the actual angle θ between e and the curve element e^* in S_{2n-1} such that e and e^* are on the same two-dimensional isocline.

In the next and final section, we shall show that the pseudo-angle characterizes the pseudo-conformal group G. It is remarked that this is a direct generalization of the fact that the group of functions of a single complex variable is identical with the conformal group of the plane.

8. **Characterization of the pseudo-conformal group G by the pseudo-angle.** We shall prove the following fundamental theorem which is essentially Kasner's characterization in $2n$ dimensions.

THEOREM 4. *A transformation of $2n$-dimensional euclidean space R_{2n} is pseudo-conformal if and only if it preserves the pseudo-angle defined (in cartesian coordinates) by*

$$\theta = \operatorname{arc\,tan} \frac{\sum_{\alpha=1}^{n} (F_{x_\alpha} dx_\alpha + F_{y_\alpha} dy_\alpha)}{\sum_{\alpha=1}^{n} (F_{x_\alpha} dy_\alpha - F_{y_\alpha} dx_\alpha)}, \tag{20}$$

between any curve $C: x_\alpha = x_\alpha(t)$, $y_\alpha = y_\alpha(t)$, and any $(2n-1)$-dimensional hypersurface $S_{(2n-1)}: F(x_1, x_2, \cdots, x_n; y_1, y_2, \cdots, y_n) = 0$ at their common point of intersection.

Any arbitrary transformation T with nonvanishing jacobian induces at any given point a general projectivity in $\Sigma_{(2n-1)}$ which in minimal curve element coordinates may be written as

$$\rho \Phi_\alpha = \sum_{\beta=1}^{n} (a_{\alpha\beta}\phi_\beta + b'_{\alpha\beta}\psi_\beta), \qquad \rho \Psi_\alpha = \sum_{\beta=1}^{n} (b_{\alpha\beta}\psi_\beta + a'_{\alpha\beta}\phi_\beta). \tag{21}$$

Also this projectivity in hypersurface element coordinates may be written as

$$(22)\qquad \sigma k_\beta = \sum_{\alpha=1}^{n} (a_{\alpha\beta}K_\alpha + a'_{\alpha\beta}L_\alpha), \qquad \sigma l_\beta = \sum_{\alpha=1}^{n} (b_{\alpha\beta}L_\alpha + b'_{\alpha\beta}K_\alpha),$$

where the determinants of the coefficients are not zero.

Now let T preserve the pseudo-angle (20). In minimal coordinates, this may be written in the form (19). Then the fraction of (19) is equal to the same fraction with the small letters replaced by the capital letters. Upon eliminating Φ_α, Ψ_α, k_β, l_β by means of (21) and (22) in this resulting identity, we find

$$(23)\qquad \frac{\sum_{\alpha\,\beta=1}^{n} L_\alpha(b_{\alpha\beta}\psi_\beta + a'_{\alpha\beta}\phi_\beta)}{\sum_{\alpha,\beta=1}^{n} K_\alpha(a_{\alpha\beta}\phi_\beta + b'_{\alpha\beta}\psi_\beta)} = \frac{\sum_{\gamma\,\delta=1}^{n} \psi_\delta(b_{\gamma\delta}L_\gamma + b'_{\gamma\delta}K_\gamma)}{\sum_{\gamma,\delta=1}^{n} \phi_\delta(a_{\gamma\delta}K_\gamma + a'_{\gamma\delta}L_\gamma)}.$$

Now this must be an identity for all $(\phi_\alpha, \psi_\alpha, K_\alpha, L_\alpha)$. Placing the coefficients of ϕ_β^2 and ψ_β^2 equal to zero, we find

$$(24)\qquad \begin{aligned} \left[\sum_{\alpha=1}^{n} a'_{\alpha\beta}L_\alpha\right]\left[\sum_{\gamma=1}^{n} (a_{\gamma\beta}K_\gamma + a'_{\gamma\beta}L_\gamma)\right] &= 0, \\ \left[\sum_{\alpha=1}^{n} b'_{\alpha\beta}K_\alpha\right]\left[\sum_{\gamma=1}^{n} (b_{\gamma\beta}L_\gamma + b'_{\gamma\beta}K_\gamma)\right] &= 0. \end{aligned}$$

These are identities in K and L. Now the second factors of each of these equations cannot identically vanish for then the determinant of (21) or (22) is zero. Therefore the preceding equations will be identities if and only if the first factors are identically zero. Therefore $a'_{\alpha\beta} = b'_{\alpha\beta} = 0$ for all $\alpha, \beta = 1, 2, \cdots, n$.

Since $a'_{\alpha\beta} = b'_{\alpha\beta} = 0$, our transformation (21) or (22) is induced pseudo-conformal. Our Theorem 4 is therefore proved, for this result is valid at any fixed point.

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