A DIFFERENTIAL INEQUALITY

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The following theorem was discovered by S. B. Jackson in connection with a problem in differential geometry.

THEOREM 1. *If* $f(x)$ *is of class* C^2 *in* $(0, a)$, $f(0)=f'(0)=0$, *and* $f''(x)\leqq K|f'(x)|+L|f(x)|$ *in* $(0, a)$, *where* K *and* L *are constants, then* $f(x)\leqq 0$ *in some interval* $(0, b)$.

If $f(x)$ is in addition analytic at 0, Theorem 1 becomes trivial, since if a_n is the first nonvanishing coefficient of the power series of $f(x)$, $x^{2-n}f''(x)$ approaches a nonzero limit, while $x^{2-n}f'(x)$ and $x^{2-n}f(x)$ approach zero, as $x\to 0$, and consequently $a_n<0$.

I shall prove the following more general theorem.

THEOREM 2. *If* $f'(x)$ *is absolutely continuous in* $(0, a)$, $f(0)=f'(0)=0$, *and*

$$f''(x) \leqq K(x)\,|f'(x)| + x^{-1}L(x)\,|f(x)| \tag{1}$$

almost everywhere in $(0, a)$, *where* $K(x)$ *and* $L(x)$ *are non-negative and integrable in* $(0, a)$, *then either* $f(x)\equiv 0$ *in some interval* $(0, b)$, *or* $f'(x)<0$ *in* $0<x<\min(a, c)$, *where* c *is such that*

$$\int_0^x \{K(t) + L(t)\}\,dt < 1, \qquad 0 < x < c. \tag{2}$$

Since $f(0)=0$, $f(x)$ is negative in $(0, c)$ when $f'(x)<0$ in $(0, c)$. Theorem 1 is contained in the special case $K(t)=K$, $L(t)=Lt$.

Theorem 2 is the best possible result of its kind; for, if $\int_0^x K(t)dt$ diverges and $K(t)$ is positive and continuous in $t>0$, the function $f(x)$ defined by $f'(x)=1/\int_x^1 K(t)dt$, $f'(0)=f(0)=0$, is positive in $x>0$ and satisfies (1) for all x such that $\int_x^1 K(t)dt>1$.

Assume that $f(x)$ is not identically zero in any interval $(0, b)$, and write $M(x)=\max_{0\leqq t\leqq x}|f'(t)|$, so that $M(x)>0$ for $x>0$ and $M(x)$ is nondecreasing.

We observe that for $0\leqq x\leqq a$,

$$|f(x)| = \left|\int_0^x f'(t)dt\right| \leqq xM(x). \tag{3}$$

There are points x_n such that $x_n \downarrow 0$ and $M(x_n)=|f'(x_n)|$. Suppose that $f'(x_n)>0$ for some n. Let (a_n, b_n) be the largest interval, contain-

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ing x_n, in which $f'(x)>0$; since $f'(x)$ is continuous, there is such an interval, and $f'(a_n)=0$. Consequently we have, using (1) and (3),

$$\begin{aligned} 0 < M(x_n) &= f'(x_n) = \int_{a_n}^{x_n} f''(t)dt \\ &\leqq \int_{a_n}^{x_n} \{K(t)f'(t) + t^{-1}L(t)\,|\,f(t)\,|\,\}dt \\ &\leqq M(x_n)\int_0^{x_n} \{K(t) + L(t)\}dt. \end{aligned}$$

This leads to a contradiction if $x_n<c$, where c is defined by (2). Hence we must have $f'(x_n)<0$ for $x_n<c$.

There is a largest interval (a_n, b_n), containing x_n, in which $f'(x)<0$. Suppose first that $a_n>0$ for every n; then the intervals (a_n, b_n) are separated by other intervals in which $f'(x)\geqq 0$, and consequently $f'(b_n)=0$, $b_n\rightarrow 0$. Then we have for $a_n<x<b_n$

$$\begin{aligned} 0 < -f'(x) &= \int_x^{b_n} f''(t)dt \\ &\leqq \int_x^{b_n} \{-K(t)f'(t) + t^{-1}L(t)\,|\,f(t)\,|\,\}dt \qquad (4) \\ &\leqq M(b_n)\int_0^{b_n} \{K(t) + L(t)\}dt. \end{aligned}$$

Since $f'(b_n)=0$, there is a point x_n' in (x_n, b_n) such that $-f'(x_n') = M(b_n)$. Taking $x=x_n'$ in (4), we have

$$0 < M(b_n) \leqq M(b_n)\int_0^{b_n} \{K(t) + L(t)\}dt, \qquad (5)$$

and again there is a contradiction when $b_n<c$.

Hence $a_n=0$ for some n. If $b_n\geqq a$, we have $f'(x)<0$ in $0<x<a$; if $b_n<a$, we have $f'(b_n)=0$, and then (4) and (5) hold. But (5) is contradictory unless $b_n\geqq c$. Hence we have $f'(x)<0$ in $0<x<\min(a, c)$.

We have incidentally established the following result about a function and its first derivative.

THEOREM 3. *If $f(x)$ is absolutely continuous in $(0, a)$, $f(0)=0$, and $f'(x)\leqq K(x)|f(x)|$, where $K(x)$ is non-negative and integrable in $(0, a)$, then either $f(x)\equiv 0$ in some interval $(0, b)$, or $f(x)<0$ in $0<x<\min(a, c)$, where $\int_0^x K(t)dt<1$ if $x<c$.*

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