A GENERALIZATION OF CONTINUED FRACTIONS[1]

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1. **Introduction.**[2] The generalizations and analogues of regular continued fractions due to Pierce [8], Lehmer [5], and Leighton [6] concern the iteration of rational functions to obtain rational approximations to a real number. The present generalization proceeds from the fact that the continued fraction

$$\frac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} \tag{1.1}$$

can be written in the form

$$f(a_1 + f(a_2 + \cdots \tag{1.2}$$

where $f(t) = 1/t$. This suggests the possibility of using functions other than $1/t$ to obtain generalizations of (1.1). In §2 a class F of functions which includes $1/t$ is defined and in §3 meaning is given to (1.2) for each $f \in F$ and each sequence $a_1, a_2, a_3, \cdots$ of positive integers. An algorithm is given for obtaining for a fixed $f \in F$ an expression of the form (1.2) corresponding to each number x in the interval $0 < x < 1$; this expression is then called the *f-expansion of x*. The analogue of the nth convergent of a simple continued fraction is defined, and its behavior with respect to x is noted. In §4 the form (1.2) is called an *f-expansion* when $f \in F$ and $a_1, a_2, a_3, \cdots$ is a sequence of positive integers. The convergence and some idea of the rapidity of convergence of an f-expansion are established. The one-to-one correspondence between f-expansions and f-expansions of numbers x, $0 < x < 1$, is given in §5 by Theorem 5. In §6 statistical independence of the a_i of an f-expansion is defined in the customary way and a subclass F_p of F for which the a_i are statistically independent is considered. Various sets of numbers x whose f-expansions are restricted by conditions on the a_i are considered and the linear Lebesgue measures of these sets are given. In §7, when $f \in F_p$, certain sets of numbers x which have been studied for $f(t) = 1/t$ by Borel [2] and F. Bernstein [1] are shown to be of measure zero.

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[2] Numbers in brackets refer to the bibliography.

2. **The class F.** Let F denote the class of real functions $f(t)$ defined for $t \geqq 1$ and having the following properties:

$$f(1) = 1; \tag{2.1}$$

$$f(t_1) > f(t_2) > 0, \qquad 1 \leqq t_1 < t_2; \tag{2.2}$$

$$\lim_{t \to \infty} f(t) = 0; \tag{2.3}$$

$$|f(t_2) - f(t_1)| < |t_2 - t_1|, \qquad 1 \leqq t_1 < t_2; \tag{2.4}$$

there is a constant λ such that $0 < \lambda < 1$ and

$$|f(t_2) - f(t_1)| < \lambda^2 |t_2 - t_1|, \qquad 1 + f(2) < t_1 < t_2. \tag{2.5}$$

3. **The f-expansions of numbers.** Let $f(t) \in F$ and x be a fixed number, $0 < x < 1$. Let z_0 be defined by $x = f(z_0)$ and let the sequences $z_1, z_2, \cdots, \theta_1, \theta_2, \cdots$, and $a_1, a_2, \cdots$ be defined by the relations

$$a_n = [z_{n-1}], \qquad \theta_n = z_{n-1} - a_n, \qquad \theta_n = f(z_n), \tag{3.1}$$

for $n = 1, 2, \cdots$. If $\theta_n \neq 0$ for $n < k$ while $\theta_k = 0$, we shall say the expansion terminates and that the *f-expansion of x* is[8]

$$f(a_1 + f(a_2 + \cdots + f(a_k). \tag{3.2}$$

In this case it is easy to see that $a_k \geqq 2$ and that the f-expansion of x is equal to x. If $\theta_n \neq 0$ for $n = 1, 2, \cdots$, then the expansion will not terminate and we shall call

$$f(a_1 + f(a_2 + \cdots \tag{3.3}$$

the *f-expansion of x*.

By analogy with simple continued fractions we define

$$x_n = f(a_1 + f(a_2 + \cdots + f(a_n) \tag{3.4}$$

and call the elements of the sequence $x_1, x_2, \cdots$ the *convergents* of x. The integers $a_1, a_2, \cdots$ and the convergents $x_1, x_2, \cdots$ are uniquely determined by x for almost all x, $0 < x < 1$. When we wish to emphasize this functional dependence we shall write them in the form $a_1(x), a_2(x), \cdots$ and $x_1(x), x_2(x), \cdots$.

To facilitate notation we introduce the function $\phi_n(t)$ defined when $f \in F$ and $a_1, a_2, \cdots$ is a sequence of positive integers by

$$\phi_n(t) = f(a_1 + f(a_2 + \cdots + f(a_n + t), \qquad t \geqq 0. \tag{3.5}$$

A simple induction proves the following lemma.

[8] In (3.2) and similar expressions we shall use a single parenthesis on the right.

LEMMA 1. *The function* $\phi_n(t)$ *is a decreasing (increasing) function of* t *when* n *is odd (even).*

THEOREM 1. *If* $f \in F$ *and* $0 < x < 1$, *then the odd (even) convergents of the* f*-expansion of* x *form a decreasing (increasing) sequence bounded below (above) by* x; *thus*

$$(3.6) \qquad 0 < x_2 < x_4 < \cdots \leqq x \leqq \cdots < x_3 < x_1 \leqq 1.$$

When $\phi_n(t)$ is defined by (3.5), we have $x_n = \phi_n(0)$, $x = \phi_n(\theta_n)$, and $x_{n+1} = \phi_n(f(a_{n+1}))$. Since $f(a_{n+1}) \geqq \theta_n > 0$, we can apply Lemma 1 to obtain $x_n > x \geqq x_{n+1}$ when n is odd and $x_n < x \leqq x_{n+1}$ when n is even. Since $f(a_{n+1} + f(a_{n+2})) > 0$ and $x_{n+2} = \phi_n(f(a_{n+1} + f(a_{n+2})))$, we similarly have $x_n > x_{n+2}$ when n is odd and $x_n < x_{n+2}$ when n is even. These inequalities establish Theorem 1.

COROLLARY. *If* $\lim_{n\to\infty} x_n$ *exists, then* $\lim_{n\to\infty} x_n = x$.

4. **Convergence of f-expansions.** If $f \in F$ we shall mean by an *f-expansion* either a finite expansion $f(a_1 + f(a_2 + \cdots + f(a_k)$ in which the a_i are positive integers and $a_k \geqq 2$, or an infinite expansion $f(a_1 + f(a_2 + \cdots$ in which the a_i are positive integers. It is to be proved later that each f-expansion is generated by a unique x; meanwhile this is not assumed.

THEOREM 2. *Let* $f \in F$. *If sequences* x_n *and* y_n *are defined in terms of an* f*-expansion by the formulas*

$$(4.1) \qquad x_n = f(a_1 + f(a_2 + \cdots + f(a_n),$$

$$(4.2) \qquad y_n = f(a_1 + f(a_2 + \cdots + f(a_n + 1),$$

then

$$(4.3) \qquad 0 < x_2 < x_4 < \cdots < x_3 < x_1 \leqq 1$$

and

$$(4.4) \qquad x_{n+1} \in I(a_1, a_2, \cdots, a_n),$$

where $I(a_1, a_2, \cdots, a_n)$ *is the closed interval with end points at* x_n *and* y_n.

Proof of (4.3) is identical with a part of the proof of (3.6). The conclusion (4.4) follows from Lemma 1 since $x_n = \phi_n(0)$,

$$x_{n+1} = \phi_n(f(a_{n+1})), \qquad y_n = \phi_n(1), \quad \text{and} \quad 0 < f(a_{n+1}) \leqq 1.$$

LEMMA 2.[4] *Let* $f \in F$. *For a fixed positive integer* n, *the least upper*

[4] We use the symbol $|E|$ to denote the linear Lebesgue measure of a set E.

bound of $|I(a_1, a_2, \cdots, a_n)|$ *for all sequences of positive integers* a_i *is less than* λ^{n-2} *where* λ *is the constant in* (2.5); *that is, if* $f \in F$ *and*

$$(4.5)\quad A_n = \operatorname*{l.u.b.}_{a_1, \cdots, a_n \geqq 1} \left| f(a_1 + \cdots + f(a_n + 1) - f(a_1 + \cdots + f(a_n) \right|,$$

where $a_1, a_2, \cdots, a_n$ *assume independently all positive integral values, then*

$$(4.6)\qquad A_n \leqq \lambda^{n-2}, \qquad n = 1, 2, \cdots.$$

For $n \geqq 1$, we can write

$$\begin{aligned} A_{n+2} &= \operatorname*{l.u.b.}_{a_1, \cdots, a_{n+2} \geqq 1} \frac{|I(a_1, \cdots, a_{n+2})|}{|I(a_3, \cdots, a_{n+2})|} \cdot |I(a_3, \cdots, a_{n+2})| \\ &\leqq A_n \cdot \operatorname*{l.u.b.}_{a_1, a_2 \geqq 1;\, 0<u<v\leqq 1} \left| \frac{f(a_1 + f(a_2 + u) - f(a_1 + f(a_2 + v)}{u - v} \right|, \end{aligned}$$

from which we obtain

$$(4.7)\quad \begin{aligned} A_{n+2} \leqq A_n \cdot \operatorname*{l.u.b.}_{a_1, a_2 \geqq 1;\, 0<u<v\leqq 1} &\left| \frac{f(a_1 + f(a_2 + u) - f(a_1 + f(a_2 + v)}{[a_1 + f(a_2 + u)] - [a_1 + f(a_2 + v)]} \right| \\ &\cdot \left| \frac{f(a_2 + u) - f(a_2 + v)}{u - v} \right|. \end{aligned}$$

If $a_2 = 1$, then $a_1 + f(a_2 + u) > a_1 + f(a_2 + v) \geqq 1 + f(2)$ when a_1 is a positive integer and $0 < u < v \leqq 1$, so that by (2.5) and (2.4) the first and second factors of the product of which the least upper bound is taken in (4.7) are less than λ^2 and 1, respectively. If $a_2 \geqq 2 > 1 + f(2)$, then the first and second factors are less than 1 and λ^2, respectively. So we have $A_{n+2} \leqq \lambda^2 A_n$, $n = 1, 2, \cdots$. Since $A_2 \leqq A_1 < 1$, the statement (4.6) follows easily by mathematical induction.

THEOREM 3. *If* $f \in F$, *then each infinite f-expansion converges to a number* x *in the interval* $0 < x < 1$; *moreover*

$$(4.8)\qquad |x_n - x| \leqq \lambda^{n-2}, \qquad n = 1, 2, \cdots,$$

where λ *is the constant in* (2.5).

From Theorem 2 and Lemma 2 we conclude that $|x_{n+1} - x_n| \leqq \lambda^{n-2}$ for $n = 1, 2, \cdots$ and since $0 < \lambda < 1$, x_n converges to a number x which by (4.3) lies in each of the intervals from x_n to x_{n+1}. This proves (4.8).

THEOREM 4. *If* $f \in F$ *and* $0 < x < 1$, *the f-expansion of* x *converges to* x.

In the terminating case the f-expansion of x obviously equals x

and in this sense converges to x. In the non-terminating case the conclusion follows directly from Theorem 3 and the corollary to Theorem 1.

Henceforth we shall use the notation $x=f(a_1+f(a_2+\cdots$ to mean that the f-expansion on the right side converges to x.

When $f(t)=1/t$, the least upper bound of $|f(x)-f(y)|/|x-y|$ for $3/2<x<y$ is $(2/3)^2$, and so we may take $\lambda=2/3$. It follows from (4.8) that

$$|x_n(x)-x| \leqq (2/3)^{n-2}, \qquad n=1,2,\cdots. \tag{4.9}$$

From the theory of simple continued fractions we know [7, 4] that

$$|x_n(x)-x| \leqq z^{n-1}, \qquad n=1,2,\cdots, \tag{4.10}$$

where $z=(3-5^{1/2})/2$. Comparison of (4.9) and (4.10) shows that our method of obtaining estimates of the rapidity of uniform convergence of f-expansions gives, when applied to $f(t)=1/t$, an estimate which is similar in form to the stronger estimate of (4.10).

5. **Uniqueness.** In this section we establish a one-to-one correspondence between f-expansions and f-expansions of numbers x, $0<x<1$. We note, as in simple continued fractions [7, p. 22], the following lemma.

LEMMA 3. *If $f\in F$, then any two of the three equations*

$$x=f(a_1+f(a_2+\cdots, \tag{5.1}$$

$$y=f(a_n+f(a_{n+1}+\cdots, \tag{5.2}$$

$$x=f(a_1+f(a_2+\cdots+f(a_{n-1}+y) \tag{5.3}$$

implies the third, the f-expansions in (5.1) *and* (5.2) *being infinite.*

The proof of Lemma 3 is straightforward.

THEOREM 5. *If $f\in F$ and $0<x<1$, then an f-expansion which converges to x and the f-expansion of x are identical.*

If the two infinite f-expansions $f(a_1+f(a_2+\cdots$ and $f(b_1+f(b_2+\cdots$ converge to the same x, then by successively applying Lemma 3 we obtain $a_n=b_n$, $n=1,2,\cdots$. A similar argument proves that an infinite f-expansion and a finite f-expansion or two different finite f-expansions do not converge to the same x. Theorem 4 completes the proof.

6. **Statistical independence.** From (3.6) and (4.4) we see that $I(c_1,c_2,\cdots,c_i)$ except for at most its end points is identical with the

set of x, $0<x<1$, for which $a_j(x)=c_j$, $j=1, 2, \cdots, i$. More exactly we have[5]

$$
\begin{aligned}
(6.1)\qquad & E[a_j(x) = c_j; j = 1, 2, \cdots, i] \\
& \qquad = I(c_1, c_2, \cdots, c_i) - \{f(c_1 + f(c_2 + \cdots + f(c_i + 1)\}
\end{aligned}
$$

unless $i=1$ and $c_1=1$ in which case

$$
(6.2)\qquad E[a_1(x) = 1] = I(1) - \{f(1)\} - \{f(2)\}.
$$

LEMMA 4. *If $f\in F$ and $c_1, c_2, \cdots, c_n$ and $c_1', c_2', \cdots, c_n'$ are two sets of positive integers such that for at least one j, $1\leqq j\leqq n$, $c_j\neq c_j'$, then the intervals $I(c_1, c_2, \cdots, c_n)$ and $I(c_1', c_2', \cdots, c_n')$ have at most an end point in common.*

The proof of this lemma follows from (6.1) and (6.2) and from the fact that the sets $E[a_j(x)=c_j;\ j=1, 2, \cdots, n]$ and $E[a_j(x)=c_j';\ j=1, 2, \cdots, n]$ are mutually exclusive by Theorem 5.

COROLLARY. *If $f\in F$, then*

$$
I(c_1, c_2, \cdots, c_n) = \{f(c_1 + \cdots + f(c_n)\} + \sum_{j=1}^{\infty} I(c_1, c_2, \cdots, c_n, j).
$$

If $y_1, y_2, \cdots$ is a decreasing sequence of positive numbers such that $y_1=1$ and $y_n\to 0$ and $f(t)$ is the function whose graph is the polygon joining in order the points (n, y_n), $n=1, 2, \cdots$, then $f(t)\in F$. Let F_p be the class of all such polygonal functions.

THEOREM 6. *If $f\in F_p$, then for any positive integers i and k*

$$
|E[a_i(x) = k]| = f(k) - f(k + 1).
$$

By (6.1) and (6.2) we have $|E[a_1(x)=k]| = |I(k)| = f(k)-f(k+1)$. For any positive integer m, it follows from (6.1) and Lemma 4 that $|E[a_{m+1}(x)=k]| = \sum |I(b_1, b_2, \cdots, b_m, k)|$ where $\sum$ is to be taken independently over all positive integral values of $b_1, b_2, \cdots, b_m$. By the mean value theorem we have

$$
\begin{aligned}
& |E[a_{m+1}(x) = k]| \\
& = \sum |f(b_1 + \cdots + f(k + 1) - f(b_1 + \cdots + f(k)| \\
& = \sum |f(b_1) - f(b_1 + 1)|\,|f(b_2 + \cdots + f(k + 1) - f(b_2 + \cdots + f(k)| \\
& = (\sum |f(b_1) - f(b_1 + 1)|)\cdot(\sum |I(b_2, \cdots, b_m, k)|) \\
& = \sum |I(b_2, \cdots, b_m, k)| = |E[a_m(x) = k]|.
\end{aligned}
$$

[5] The symbol $E[\cdots]$ shall denote the set of x satisfying the proposition in brackets.

An induction completes the proof.

The functions $a_i(x)$, $i=1, 2, \cdots$, are said to be *statistically independent* [4] if for each set of positive integers $n_1<n_2<\cdots<n_m$ and each set of positive integers $c_1, c_2, \cdots, c_m$

$$(6.3)\quad \left| E[a_{n_j}(x) = c_j; j = 1, 2, \cdots, m] \right| = \prod_{j=1}^{m} \left| E[a_{n_j}(x) = c_j] \right|.$$

THEOREM 7. *If $f\in F_p$, then the functions $a_i(x)$, $i=1, 2, \cdots$, are statistically independent.*

The equation (6.3) is trivial for $m=1$. By (6.1) and Lemma 4 we have

$$\begin{aligned} &\left| E[a_{n_j}(x) = c_j; j = 1, 2, \cdots, m] \right| \\ &\qquad = \sum{}' \left| I(b_1, \cdots, b_{n_1-1}, c_1 b_{n_1+1}, \cdots, c_2, \cdots, c_m) \right| \end{aligned}$$

where $\sum'$ is to be taken independently over all positive integral values of b_i for all indices i from one to n_m excepting $i=n_1, n_2, \cdots, n_m$. By an argument similar to that used in the proof of Theorem 6 we obtain

$$\begin{aligned} &\left| E[a_{n_j}(x) = c_j; j = 1, 2, \cdots, m] \right| \\ &\quad = \sum{}' \left| f(b_1) - f(b_1+1) \right| \cdot \left| I(b_2, \cdots, b_{n_1-1}, c_1, \cdots, c_m) \right| \\ &\quad = \sum{}' \left| I(b_2, \cdots, b_{n_1-1}, c_1, \cdots, c_m) \right| = \cdots \\ &\quad = \sum{}' \left| I(c_1, b_{n_1+1}, \cdots, c_m) \right| \\ &\quad = \left| f(c_1) - f(c_1+1) \right| \cdot \left(\sum{}' \left| I(b_{n_1+1}, \cdots, c_m) \right| \right) \\ &\quad = \left| E[a_{n_1}(x) = c_1] \right| \cdot \left| E[a_{n_j}(x) = c_j; j = 2, \cdots, m] \right| \end{aligned}$$

and again an induction completes the proof.

COROLLARY. *If $f\in F_p$, then for each set of positive integers $n_1<n_2<\cdots<n_m$ and each set of positive integers $c_1, c_2, \cdots, c_m, d_1, d_2, \cdots, d_m$ such that $c_j\leqq d_j$, $j=1, 2, \cdots, m$, we have*

$$\begin{aligned} \left| E[c_j \leqq a_{n_j}(x) \leqq d_j; j = 1, 2, \cdots, m] \right| &= \prod_{j=1}^{m} \left| E[c_j \leqq a_{n_j}(x) \leqq d_j] \right| \\ &= \prod_{j=1}^{m} \left| f(c_j) - f(d_j+1) \right|. \end{aligned}$$

7. Sets of measure zero.[6] The results of §6 will now be used in order to prove a few measuretheoretical facts concerning f-expansions under the assumption that $f\in F_p$.

[6] Theorems, similar to those in this section, applying to the simple continued fraction have been proved by Borel [2] and Bernstein [1]; for expositions see [3].

THEOREM 8. *If $f\in F_p$, then the set of x, $0<x<1$, for which the sequence $a_1(x), a_2(x), \cdots$ is bounded, has measure zero.*

Let the set $E[a_i(x)\leqq k; i=1, 2, \cdots, m]$ be denoted by G_k^m. In the corollary to Theorem 7 we set $n_j=j$, $c_j=1$, $d_j=k$ and obtain

$$|G_k^m| = \prod_{j=1}^{m} \{1-f(k+1)\} = \{1-f(k+1)\}^m.$$

If we let $G_k=E[a_i(x)\leqq k; i=1, 2, \cdots]$, then $G_k\in G_k^m$, $m=1, 2, \cdots$, and so $|G_k|=0$. The set of x, $0<x<1$, for which the sequence $a_1(x), a_2(x), \cdots$ is bounded is $G=\sum_{i=1}^{\infty}G_i$ and consequently $|G|=0$.

Similarly the set of x, $0<x<1$, for which $a_i(x)>k$, $i=1, 2, \cdots, m$, has measure $\{f(k+1)\}^m$. An argument similar to that used in the proof of Theorem 8 proves the following theorem.

THEOREM 9. *If $f\in F_p$, then the set of x, $0<x<1$, for which $a_i(x)>1$, $i=1, 2, \cdots$, has measure zero.*

THEOREM 10. *If $f\in F_p$ and $\phi(1), \phi(2), \cdots$ is a sequence of positive integers for which*

$$\sum_{n=1}^{\infty} f(\phi(n)+1) \tag{7.1}$$

is divergent, then the set of x, $0<x<1$, for which $a_n(x)\leqq\phi(n)$, $n=1, 2, \cdots$, has measure zero.

Let $H_m=E[a_i(x)\leqq\phi(i); i=1, 2, \cdots, m]$. By an argument similar to the one used in proving Theorem 8 we have

$$|H_m| = \prod_{i=1}^{m} \{1-f(\phi(i)+1)\}. \tag{7.2}$$

Since $0<f(\phi(i)+1)<1$ for $i=1, 2, \cdots$, the divergence of the series (7.1) is equivalent to the limit as $m\to\infty$ of the product in (7.2) being zero. If we let $H=E[a_i(x)\leqq\phi(i); i=1, 2, \cdots]$, then since $H\in H_m$ for every positive integer m, it follows that $|H|=0$.

The last three theorems can be generalized to infinite subsequences of the sequence $a_1(x), a_2(x), \cdots$.

BIBLIOGRAPHY

1. F. Bernstein, *Über eine Anwendung der Mengenlehre auf ein aus der Theorie der säkularen Störungen herrührendes Problem*, Math. Ann. vol. 71 (1912) pp. 417–439, especially §2.

2. E. Borel, *Les probabilités dénombrables et leurs applications arithmétiques*, Rend. Circ. Mat. Palermo vol. 27 (1909) pp. 246–271.

3. G. H. Hardy and E. M. Wright, *Introduction to the theory of numbers*, Oxford, 1938, chaps. 10–11.

4. M. Kac, *Sur les fonctions indépendantes*, I, Studia Math. vol. 6 (1936) pp. 46–58.

5. D. H. Lehmer, *A cotangent analogue of continued fractions*, Duke Math. J. vol. 4 (1938) pp. 323–340.

6. W. Leighton, *Proper continued fractions*, Amer. Math. Monthly vol. 47 (1940) pp. 274–280.

7. O. Perron, *Die Lehre von den Kettenbrüchen*, 2nd ed., Leipzig, 1929.

8. T. A. Pierce, *On an algorithm and its use in approximating roots of an algebraic equation*, Amer. Math. Monthly vol. 36 (1929) pp. 523–525.

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