THE ROLE OF INTERNAL FAMILIES IN MEASURE THEORY

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1. **Introduction.** Theorem 4.7 below is an abstract formulation of a certain closed subset theorem[1] recently established by Randolph and myself. It has a wider range of application than similar abstractions due to Hahn[2] and to Saks.[3]

2. **Notation and terminology.** When H is a family of sets we agree that

$$\sigma(H) = \sum_{\beta \in H} \beta, \qquad \pi(H) = \prod_{\beta \in H} \beta.$$

A family R is said to be: *finitely additive* if $\sigma(H) \in R$ whenever H is a finite nonvacuous subfamily of R; *countably additive* if $\sigma(H) \in R$ whenever H is a countable nonvacuous subfamily of R; *finitely multiplicative* if $\pi(H) \in R$ whenever H is a finite nonvacuous subfamily of R; *countably multiplicative* if $\pi(F) \in R$ whenever F is a countable nonvacuous subfamily of R; α *complemental* if R is such a family of subsets of α that $\alpha - \beta \in R$ whenever $\beta \in R$.

If R is a family of sets we also agree that: R_σ is the family of all sets of the form $\sigma(H)$ where H is a countable nonvacuous subfamily of R; R_δ is the family of all sets of the form $\pi(H)$ where H is a countable nonvacuous subfamily of R; R_γ is the family of all sets of the form $\sigma(R) - \beta$ where $\beta \in R$; R^γ is the smallest $\sigma(R)$ complemental, countably additive family which contains R; R^δ is the smallest countably multiplicative, countably additive family which contains R.

Definition 2.1. R is *internal* if and only if R_δ is finitely additive and $R_\gamma \subset R^\delta$.

Remark 2.2 If R is the family of all closed subsets of a metric space then R is internal[4] and the members of R^γ are the Borel subsets of the space.

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[1] A. P. Morse and J. F. Randolph, *The ϕ rectifiable subsets of the plane*, Trans. Amer. Math. Soc. vol. 55 (1944) pp. 236–305, Theorem 3.7 together with the remarks which follow Theorem 3.4.

[2] H. Hahn, *Über die Multiplikation total-additiver Mengenfunktionen*, Annali della R. Scuola Normale Superiore Pisa (2) vol. 2 (1933) p. 437.

[3] S. Saks, *Theory of the integral*, Warsaw, 1937, p. 85.

[4] Since an open set is an R_σ.

3. Two known results in set theory.

THEOREM 3.1. *R_δ is countably multiplicative. If R is finitely additive then so is R_δ.*

PROOF. R_δ is clearly countably multiplicative. The remainder of the theorem follows from the identity

$$\prod_{x\in A} x + \prod_{y\in B} y = \prod_{x\in A}\ \prod_{y\in B} (x+y).$$

THEOREM 3.2.[5] *If $R_\gamma \subset R^\delta$ then $R^\gamma = R^\delta$.*

PROOF. Let $\tilde{\alpha} = \sigma(R) - \alpha$. Let

$$P = \underset{\alpha}{E}\, [(\alpha \in R^\delta)(\tilde{\alpha} \in R^\delta)].$$

A simple check reveals that P is a $\sigma(R)$ complemental, countably additive subfamily of R^δ. Our assumption that R_γ is contained in R^δ assures us, on the other hand, that P contains R. Accordingly $R^\gamma \subset P \subset R^\delta$. Now R^γ, being $\sigma(R)$ complemental and countably additive, is clearly countably multiplicative also. Consequently $R^\delta \subset R^\gamma$ and the desired conclusion is at hand.

4. The role of internal families in measure theory.

DEFINITION 4.1. We say *ϕ measures S* if and only if ϕ is such a function on $E_\beta[\beta \subset S]$ to $E_t[0 \leqq t \leqq \infty]$ that:

I. $\phi(0) = 0$;

II. $\phi(A) \leqq \phi(B)$ whenever $A \subset B \subset S$;

III. If H is any countable family for which $\sigma(H) \subset S$, then

$$\phi[\sigma(H)] \leqq \sum_{\beta\in H} \phi(\beta).$$

THEOREM 4.2. *If ϕ measures S and ϕ measures T then $S = T$.*

Due to Carathéodory[6] is

DEFINITION 4.3. A set A is *ϕ measurable* if and only if ϕ measures some superset S of A in such a way that

$$\phi(T) = \phi(TA) + \phi(T - A)$$

whenever $T \subset S$.

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[5] This is a corollary of a theorem proved by W. Sierpinski in his *Les ensembles boreliens abstraits*, Annales de la Société polonaise de mathématique vol. 6 (1927) p. 51.

[6] C. Carathéodory, *Über das lineare mass von Punktmengen*, Nachr. Ges. Wiss. Göttingen (1914) p. 406.

THEOREM 4.4. *If R is a family of ϕ measurable sets, ϕ measures $\sigma(R)$, then R^δ and R^γ are families of ϕ measurable sets.*

PROOF. Let M be the family of all ϕ measurable sets. Clearly M is $\sigma(R)$ complemental and countably additive.[7] Consequently $R^\delta \subset R^\gamma \subset M$.

LEMMA 4.5. *If R_δ is a finitely additive family of ϕ measurable sets, ϕ measures $\sigma(R)$, $\phi[\sigma(R)]<\infty$, $B\in R^\delta$, $\epsilon>0$, then B contains such a member C of R_δ that $\phi(B-C)<\epsilon$.*

PROOF. Let K be so defined that $\beta\in K$ if and only if corresponding to each positive number η there is such a member C of R_δ that

$$C \subset \beta, \qquad \phi(\beta - C) < \eta.$$

We shall complete the proof by showing in Part III below that $B\in K$.

Part I. *If H is a countable nonvacuous subfamily of K then $\sigma(H)\in K$ and $\pi(H)\in K$.*

PROOF. Let $\eta>0$. Let $A_1, A_2, A_3, \cdots$ be a sequence whose range is H. Let $C_1, C_2, C_3, \cdots$ be such members of R_δ that

$$C_n \subset A_n, \qquad \phi(A_n - C_n) < \frac{\eta}{2^n}$$

for each positive integer n.

Now

$$\phi\left[\sigma(H) - \sum_{n=1}^{\infty} C_n\right] = \phi\left[\sum_{n=1}^{\infty} A_n - \sum_{n=1}^{\infty} C_n\right] \leqq \phi\left[\sum_{n=1}^{\infty}(A_n - C_n)\right]$$
$$\leqq \sum_{n=1}^{\infty}\phi(A_n - C_n) < \sum_{n=1}^{\infty}\frac{\eta}{2^n} = \eta.$$

Accordingly if N is a sufficiently large integer we are sure that

$$\sum_{n=1}^{N} C_n \in R_\delta, \qquad \sum_{n=1}^{N} C_n \subset \sigma(H), \qquad \phi\left[\sigma(H) - \sum_{n=1}^{N} C_n\right] < \eta.$$

On the other hand

$$\pi(H) = \prod_{n=1}^{\infty} A_n$$

[7] Those measure theoretic results of which we assume a previous knowledge are in H. Hahn, *Theorie der reellen Funktionen*, vol. 1, Berlin, 1921, pp. 424–427.

and $\prod_{n=1}^{\infty} C_n$ is such a member (see 3.1) of R_δ that

$$\prod_{n=1}^{\infty} C_n \subset \pi(H),$$

$$\phi\left[\pi(H) - \prod_{n=1}^{\infty} C_n\right] = \phi\left[\sum_{n=1}^{\infty} \{\pi(H) - C_n\}\right] \leqq \phi\left[\sum_{n=1}^{\infty} (A_n - C_n)\right]$$

$$\leqq \sum_{n=1}^{\infty} \phi(A_n - C_n) < \sum_{n=1}^{\infty} \frac{\eta}{2^n} = \eta.$$

Part II. $R \subset K$.

PROOF. $R \subset R_\delta \subset K$.

Part III. $B \in K$.

PROOF. Parts I and II assure us that K is a countably multiplicative, countably additive family which contains R. Consequently $R^\delta \subset K$ and the conclusion that $B \in K$ follows from our hypothesis that $B \in R^\delta$.

THEOREM 4.6. *If R_δ is a finitely additive family of ϕ measurable sets, ϕ measures $\sigma(R)$, $B \in R^\delta$, $\phi(B) < \infty$, $\epsilon > 0$, then B contains such a member C of R_δ that $\phi(B - C) < \epsilon$.*

PROOF. Let Φ be such a function on the subsets of $\sigma(R)$ that

$$\Phi(\alpha) = \phi(B\alpha) \quad \text{whenever} \quad \alpha \subset \sigma(R).$$

Check that Φ measures $\sigma(R)$ and that 4.5 may be applied to yield the desired conclusion.

THEOREM 4.7. *If R is an internal family of ϕ measurable sets, ϕ measures $\sigma(R)$, $B \in R^\gamma$, $\phi(B) < \infty$, $\epsilon > 0$, then B contains such a member C of R_δ that $\phi(B - C) < \epsilon$.*

PROOF. Use 4.6, 2.1, and 3.2.

DEFINITION 4.8. We say ϕ is a *Borelian measure* with respect to R if and only if: R is an internal family of ϕ measurable sets; ϕ measures $\sigma(R)$; corresponding to each subset A of $\sigma(R)$ there is a set β for which

$$\beta \in R^\gamma, \qquad A \subset \beta, \qquad \phi(A) = \phi(\beta).$$

THEOREM 4.9. *If ϕ is a Borelian measure with respect to R, A is a ϕ measurable set, $\phi(A) < \infty$, $\epsilon > 0$, then A contains such a member C of R_δ that $\phi(A - C) < \epsilon$.*

PROOF. Let B', B'', B''' be such sets that

$$A \subset B' \in R^\gamma, \qquad \phi(B') = \phi(A),$$

$$B' - A \subset B'' \in R^\gamma, \qquad \phi(B'') = \phi(B' - A),$$
$$B''' = B' - B''.$$

Clearly

$$B''' \in R^\gamma, \qquad B''' = B' - B'' \subset B' - (B' - A) \subset A,$$
$$\phi(A - B''') \leqq \phi(B' - B''') \leqq \phi(B'') = \phi(B') - \phi(A) = 0.$$

Application of 4.7 to the set B''' completes the proof.

THEOREM 4.10. *If R is the family of all closed subsets of a metric space S, ϕ measures S in such a way that closed sets are ϕ measurable, B is a Borel set, $\phi(B) < \infty$, $\epsilon > 0$, then B contains such a closed set C that $\phi(B - C) < \epsilon$.*

PROOF. Clearly R is an internal family for which $R = R_\delta$, and $\sigma(R) = S$. Application of 4.7 completes the proof. Using 4.9 we obtain

THEOREM 4.11. *If R is the family of all closed subsets of a metric space S, ϕ is a Borelian measure with respect to R, A is ϕ measurable, $\phi(A) < \infty$, $\epsilon > 0$, then A contains such a closed set C that $\phi(A - C) < \epsilon$.*

REMARK 4.12. Theorems 4.9 and 4.11 are generalizations of a result due to Hahn.[8] For corollaries and special cases of Theorems 4.7, 4.9, 4.10, and 4.11, see Saks, op. cit., Theorem 6.5 on page 68, Theorem 6.6 on page 69, the correct portions of Theorem 9.7+ on page 85, the proof of Lemma 5.1 on page 114, Lemma 15.1 on page 152.

Let us now examine, in the light of an example, the just cited Theorem 9.7+ and my own Theorem 4.7. Let S be the ordinary real numbers metrized in the customary manner. Let F be the family of all closed subsets of S, G the family of all open subsets of S. Let $R = F_\sigma G_\delta$. It is easily seen, with the aid of 3.1, that R is a finitely additive, S complemental, internal family. Furthermore $\sigma(R) = S$ and R^γ is precisely the family of all Borel subsets of S. Let B be the rational numbers and let ϕ so measure S that

$$\phi(\beta) = \text{the number of numbers in } \beta B$$

whenever $\beta \subset S$. Note that $\phi(B) = \phi(S) = \infty$ but that S is a countable sum of Borel sets of finite ϕ measure. However, within the Borel set B, it is impossible to find a G_δ set C for which $\phi(B - C) < 1$; if this could be done then C would equal B and B itself would be a G_δ in contradiction to the well known fact that a dense G_δ is a residual set with the power of the continuum. Since $R_\delta \subset G_\delta$ it is also impossible to find, within the Borel set B, an R_δ set C for which $\phi(B - C) < 1$.

[8] H. Hahn, *Theorie der reellen Funktionen*, vol. 1, Berlin, 1921, p. 447, Theorem IV.

This reveals the essential nature of the hypothesis "$\phi(B) < \infty$" in 4.7 as well as the erroneous aspects of the "more generally" part of Saks' Theorem 9.7+. Nevertheless it is easy to verify the statement obtained from Theorem 4.10 by deleting the hypothesis "$\phi(B) < \infty$" and replacing it by "each bounded set has finite ϕ measure."

REMARK 4.13. Herein we give a supplementary example which serves much the same purpose as the one just discussed in 4.12. Let S be the plane metrized in the customary manner. Introduce F, G, and R as in 4.12. Let B be those points in the plane whose first coordinates are rational. Let ϕ so measure S that

$$\phi(\beta) = \text{the Carathéodory}^9 \text{ linear measure of } \beta B$$

whenever $\beta \subset S$. Note that $\phi(B) = \phi(S) = \infty$ but that S is a countable sum of Borel sets of finite ϕ measure. Note also (cf. 4.12) that each countable subset of S has ϕ measure zero. However, within the Borel set B, it is impossible to find a G_δ set C for which $\phi(B-C) < \infty$. To see this use the fact that the projection upon the y axis of any subset α of B has a Lebesgue measure which does not exceed $\phi(\alpha)$, and then recall the reasoning employed in 4.12.

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[9] C. Carathéodory, op. cit., pp. 420 ff.