ON ROTATION GROUPS OF PLANE CONTINUOUS CURVES UNDER POINTWISE PERIODIC HOMEOMORPHISMS

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In this paper we make use of the work of G. T. Whyburn[1] on light interior transformations and on orbit decompositions of certain spaces to obtain a theorem by means of which a certain subset of the orbits of points under a periodic transformation $T(M)=M$ may be given a linear ordering. This theorem is then used to obtain an accessibility theorem for plane continuous curves similar to one previously published by L. Whyburn.[2] We take this opportunity to express our indebtedness to G. E. Schweigert for suggesting the proof of Theorem I given below and thus eliminating the longer and less interesting proof previously obtained by the author. For any $x \in M$, the orbit of x under T means $O(x)=\sum_{i=-\infty}^{\infty} T^i(x)$.

THEOREM I. *Let M be a locally connected continuum (that is, a continuous curve) and $T(M)=M$ an arbitrary periodic homeomorphism. Then if a and b are arbitrary points of M lying in different orbits under T and if axb is any simple arc in M joining a and b, then there must exist a simple arc $a'x'b'$ in M lying in the orbit of axb under T such that a' belongs to $O(a)$, b' belongs to $O(b)$ and no two points of $a'x'b'$ lie in the same orbit under T. Furthermore, the point a' may be any arbitrary preassigned point of the orbit of a.*

Proof (Schweigert). Let M' be the hyperspace obtained by decomposing the space M into its orbits under T. Then, since the orbit decomposition is continuous,[3] it follows[4] that there exists a light interior transformation $f(M)=M'$, namely, the transformation given by and associated with the orbit decomposition. Let axb be the given arc in M. Then we may assume without loss of generality that axb has precisely the point a in common with $O(a)$ and precisely the point b in common with $O(b)$. Define $K=f(axb)$. Then K is a locally connected continuum containing $c=f(a)$ and $d=f(b)$. Let cyd be an arc in K joining c to d. Now let a' be an arbitrary point of $O(a)$. Then[5]

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[1] See G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloquium Publications, vol. 28, 1942, pp. 182–189 and 239–262.

[2] See L. Whyburn, *Rotation groups about a set of fixed points*, Fund. Math. vol. 28 (1937) pp. 124–130, in particular p. 127.

[3] See G. T. Whyburn, loc. cit. p. 258.

[4] See G. T. Whyburn, loc. cit. p. 130.

[5] See G. T. Whyburn, loc. cit. p. 186.

there must exist a simple arc $a'x'b'$ in M such that $f(a'x'b')=cyd$ is topological. By definition of f we see that each point of $a'x'b'$ belongs to the orbit of some point of axb, and from the one-to-oneness of this transformation it is immediate that no two points of $a'x'b'$ lie in the same orbit under this transformation. This completes the proof.

COROLLARY. *The same conclusion holds for any pointwise periodic $T(M)=M$ if we impose the additional restriction either that T have equicontinuous powers*[6] *or that the period function remain bounded on the arc axb.*

The accessibility theorem for plane continuous curves mentioned in the introductory paragraph of this paper may be stated as follows.

THEOREM A (L. WHYBURN). *If M is a plane continuous curve and $T(M)=M$ is a homeomorphism and if C is an element of a rotation group of M under T of order at least two, then C has property S.*[7]

The object of our second theorem is to obtain a result similar to Theorem A, but with the emphasis in the hypothesis placed upon the type of the transformation T rather than upon the order of the rotation group under consideration. Before stating the theorem we recall certain important subsets of M. By L we denote the closed invariant subset of M consisting of those points at which the period function has an unbounded limit superior. By K we denote the collection of all fixed points of M under T. If R is a component of $M-K$, then R is an element of a rotation group under T; this rotation group consists exactly of the orbit of R under T; and its order is the number of components which it contains. The order of a rotation group under T may, of course, be either finite or infinite.

We are now in a position to state our second theorem.

THEOREM II. *Let M be a plane continuous curve and $T(M)=M$ an arbitrary homeomorphism, while R denotes an element of some rotation group of M under T. Then*

(a) *If K is locally connected then R has property S and every point of $F(R)=\overline{R}-R$ is regularly accessible*[8] *from R.*

(b) *If T is pointwise periodic and has equicontinuous powers we get the same conclusion as in* (a).

(c) *If T is pointwise periodic then every point of $F(R)$ which is not a point of L is regularly accessible from R.*

Proof. Note that (a) is immediate from a theorem of G. T. Why-

[6] See G. T. Whuburn, loc. cit. p. 258.

[7] See G. T. Whyburn, loc. cit. p. 20.

[8] See G. T. Whyburn, loc. cit. p. 111.

burn,[9] and that it also follows at once from the proof of Theorem A.

We give the proofs of (b) and (c) simultaneously, making use of the corollary to Theorem I in each case. It is to be noted that our proof is similar to the original proof of Theorem A.

Denote by d a positive number exceeding the diameter of the set M, and suppose that M is embedded in the upper half of the Euclidean plane. If the theorem be false there must exist a point p in $F(R)$ which is not regularly accessible from R and (unless T has equicontinuous powers) such that p does not lie in L. This means that there exists a positive number ϵ and a sequence of points $\{p_i\}$ of R converging to p such that no two points of this sequence may be joined in R by a connected set of diameter less than 17ϵ. Let C_p be a circle of radius 8ϵ having its center at the point p. No generality is lost by the following assertion:

(1) *For every i the set $O(p_i)$ lies within C_p; no two of the points p_i may be joined by a connected subset of R lying within this circle; and if T does not have equicontinuous powers then there exists an integer N such that no point of M lying within C_p has period greater than N under T.*

For some point q' of R exterior to C_p we construct arcs p_iq' in R for every integer i and we denote the first intersection of the arc p_iq' with the circle C_p by q_i. It follows from (1) that the arcs p_iq_i are pairwise disjoint and we may assume, exactly as in the proof of Theorem A, that this sequence of arcs converges to a limiting set H which is a subcontinuum of K. Making use of the corollary to Theorem I we can insure that no arc p_iq_i meets the orbit of any point of M in more than a single point, and that no two of these arcs meet the orbit of the same point of M. This means, in particular, that no two consecutive images under T of any one of these arcs will have a point in common. We may assume that the sequence $\{q_i\}$ converges monotonically on C_p to a point q.

We place the x-axis in such a position that p lies at the point $(-4\epsilon, 2d)$ and q at the point $(4\epsilon, 2d)$. By L_i $(i=\pm 1, \pm 2, \pm 3)$ we denote the line segment joining $(i\epsilon, 0)$ to $(i\epsilon, 4d)$, and by $D_{i,j}$ the interior of the rectangle formed by L_i, L_j, $y=0$, and $y=4d$.

Using the fact that H is a subset of K and either (b) or (c)[10] we may make the following assumption without loss of generality:

(2) *If x is any point of any arc p_iq_i then $O(x)$ lies within a circle having its center at some point of H and diameter $\epsilon/2$.*

Let r_is_i be a subarc of p_iq_i with its interior in $D_{-3,3}$ and its end

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[9] See G. T. Whyburn, *Concerning the open subsets of a plane continuous curve*, Proc. Nat. Acad. Sci. U.S.A. vol. 13 (1927) pp. 650–657, in particular Theorems 1 and 5 of this paper.

[10] See G. T. Whyburn, *Analytic topology*, loc. cit. p. 252.

points on L_{-3} and L_3; $r_i' s_i'$ the arc $T(r_i s_i)$; $x_i' y_i'$ a subarc of $r_i' s_i'$ with its interior in $D_{-2,2}$ and its end points on L_{-2} and L_2; and $x_i y_i$ the arc $T^{-1}(x_i' y_i')$. The existence of the arc $x_i' y_i'$ follows from (2) and by the same token we see that $x_i y_i$ is a subarc of $r_i s_i$ having its end points in the respective regions $D_{-3,-1}$, $D_{1,3}$. Now the sequence of arcs $\{x_i' y_i'\}$ may be assumed to converge to a subcontinuum H' of H which is, of course, disjoint with every one of these arcs. This means that by taking a subsequence the following assumption will hold.

(3) *For any fixed integer i and every k exceeding i the arc $x_k' y_k'$ separates $D_{-2,2}$ between $x_i' y_i'$ and $x_n' y_n'$ for every n greater than k. Thus $x_k' y_k'$ separates $D_{-2,2}$ between $x_i' y_i'$ and H' for every k exceeding i.*

For any fixed value of i we know that the closed sets $O(x_i y_i)$ and H' are disjoint, which means that there will exist a region U_i in the plane containing H' but disjoint with $O(x_i y_i)$. We may assume that U_i contains $O(x_k y_k)$ for every k exceeding i. We also know, in view of this last remark, that for i fixed either $x_i y_i$ separates $D_{-1,1}$ between $x_i' y_i'$ and every $x_k' y_k'$ for k exceeding i or $x_i' y_i'$ separates this region between $x_i y_i$ and $x_k' y_k'$ for every k exceeding i. By taking a subsequence and renumbering we can insure that the same one of these two statements holds for every value of i and thus obtain the following assertion.

(4) *If i and k be any two distinct integers then in the region $D_{-1,1}$ the four arcs $x_i y_i$, $x_i' y_i'$, $x_k y_k$, $x_k' y_k'$ must occur either in the order just specified or in the alternative order $x_i' y_i'$, $x_i y_i$, $x_k' y_k'$, $x_k y_k$.*

No generality is lost by the assumption that for every integer i the arc $x_i y_i$ has an interior point within the circle C_z, having its center at a point z on the y-axis and radius sufficiently small so that any two points of M lying within C_z may be joined by an arc of M the orbit of which lies within $D_{-1,1}$. This enables us to find a simple arc $a_i b_k$ lying in M with $O(a_i b_k)$ in $D_{-1,1}$ and having exactly the points a_i, b_k in common with $x_i y_i$ and $x_k y_k$, respectively. As the two arrangements given in (4) are symmetrical we need treat only the case of the first one; the other will follow by a simple interchange of the letters i and k. From (1) we see that the arc $a_i b_k$ must contain at least one point of the closed set K, and we denote by f_{ik} the last point of K on this arc. Then if g_k be the first point of $O(x_k y_k)$ on the arc $f_{ik} b_k$, it follows that for some integer n the point $g_k' = T^n(g_k)$ lies on the arc $x_k' y_k'$. Thus the arc $f_{ik} g_k' = T^n(f_{ik} g_k)$ is a simple arc lying in $D_{-1,1}$ and joining the point f_{ik} to a point of $x_k' y_k'$, while containing no point of $x_k y_k$. This contradiction completes the proof of Theorem II.

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