ALMOST ORTHOGONAL SERIES

RICHARD BELLMAN

1. Almost orthogonal series. Let us consider an infinite sequence $\{\phi_n(x)\}$, $n=1, 2, \cdots$, of complex-valued functions of the real variable x, of class $L^2(a, b)$, normalized so that $\int_a^b |\phi_n(x)|^2 dx = 1$ for all n. Assume further that the sequence satisfies the following condition

$$\sum_{m,n} | a_{mn} |^2 < \infty, \tag{1}$$

where $a_{mn} = \int_a^b \phi_m \bar{\phi}_n dx$ ($m \neq n$; $n, m = 1, 2, \cdots$), $a_{mn} = 0$, $m = n$.

We wish to show that under the above conditions we have a Bessel inequality and an analogue of the Riesz-Fisher theorem.

THEOREM 1 (BESSEL'S INEQUALITY). *Under the above conditions, let $f(x)$ be a real-valued function belonging to $L^2(a, b)$, and $b_n = \int_a^b f \bar{\phi}_n dx$, then*

$$\sum_1^\infty | b_k |^2 \leqq \int_a^b | f |^2 dx \left[1 + \left(\sum_{m,n} | a_{mn} |^2 \right)^{1/2} \right].$$

We have

$$\sum_1^n | b_k |^2 = \int_a^b f \left[\sum_1^n \bar{b}_k \bar{\phi}_k \right] dx.$$

Using Schwartz's inequality, this becomes

$$\begin{aligned}
\sum_1^n | b_k |^2 &\leqq \left[\int_a^b | f |^2 dx \right]^{1/2} \left[\int_a^b \left[\sum_1^n b_k \phi_k \right] \left[\sum_1^n \bar{b}_k \bar{\phi}_k \right] dx \right]^{1/2} \\
&\leqq \left[\int_a^b | f |^2 dx \right]^{1/2} \left[\sum_1^n | b_k |^2 + \sum_{1,1,k\neq l}^{n,n} b_k \bar{b}_l a_{kl} \right]^{1/2} \\
&\leqq \left[\int_a^b | f |^2 dx \right]^{1/2} \left[\sum_1^n | b_k |^2 \right. \\
&\qquad \left. + \left\{ \sum_{1,1}^{n,n} | b_k |^2 | b_l |^2 \right\}^{1/2} \left\{ \sum_{1,1}^{n,n} | a_{kl} |^2 \right\}^{1/2} \right]^{1/2} \\
&\leqq \left[\int_a^b | f |^2 dx \right]^{1/2} \left[\sum_1^n | b_k |^2 \right]^{1/2} \\
&\qquad \cdot \left[1 + \left(\sum_{m,n} | a_{kl} |^2 \right)^{1/2} \right]^{1/2}.
\end{aligned}$$

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Hence, simplifying,

$$\sum_1^n | b_k |^2 \leqq \left(\int_a^b | f |^2 dx \right) \left[1 + \left(\sum_{m,n} | a_{kl} |^2 \right)^{1/2} \right],$$

and since this is true for all n, the result is obtained.

THEOREM 2 (ANALOGUE OF RIESZ-FISHER). *Under the above conditions, if* $\sum_1^\infty | b_n |^2 < \infty$, *there exists a function* $f(x) \subset L^2(a, b)$ *such that*

$$\sum_1^\infty \left| b_k - \int_a^b f\bar{\phi}_k dx \right|^2 \leqq \left(\sum_1^\infty | b_k |^2 \right) \left(\sum_{k,l} | a_{kl} |^2 \right),$$

and therefore $\lim_{n\to\infty} (b_n - \int_a^b f\bar{\phi}_n dx) = 0$.

Let $s_n = \sum_1^n b_k \phi_k$, then

$$\begin{aligned} \int_a^b & | s_m - s_n |^2 dx \\ &= \int_a^b \left[\sum_{n+1}^m b_k \phi_k \right] \left[\sum_{n+1}^m \bar{b}_k \bar{\phi}_k \right] dx \\ &= \sum_{n+1}^m | b_k |^2 + \sum_{n+1 \leqq k, l \leqq m} b_k \bar{b}_l a_{kl} \\ &\leqq \sum_{n+1}^m | b_k |^2 + \left(\sum_{n+1}^m | b_k |^2 | b_l |^2 \right)^{1/2} \left(\sum_{m,n} | a_{mn} |^2 \right)^{1/2} \\ &\leqq \sum_{n+1}^m | b_k |^2 \left(1 + \left(\sum_{m,n} | a_{mn} |^2 \right)^{1/2} \right) \end{aligned}$$

(using Schwartz's inequality). Hence, since $\sum b_n^2 < \infty$, s_n converges in mean to a function $f(x) \subset L^2(a, b)$, that is,

$$\lim_{n\to\infty} \int_a^b | f(x) - s_n(x) |^2 dx = 0.$$

We have, $k < n$,

$$b_k - \int_a^b f\bar{\phi}_k dx = b_k - \int_a^b s_n \bar{\phi}_k dx + \int_a^b (s_n - f)\bar{\phi}_k dx,$$

$$\begin{aligned} \int_a^b | s_n - f | | \phi_k | dx &\leqq \left(\int_a^b | f - s_n |^2 dx \right)^{1/2} \left(\int_a^b | \phi_k |^2 dx \right)^{1/2} \\ &= \left(\int_a^b | f - s_n |^2 dx \right)^{1/2}, \end{aligned}$$

$$b_k - \int_a^b s_n\bar{\phi}_k dx = \sum_{j=1}^n b_j a_{kj},$$

$$\left| b_k - \int_a^b s_n\bar{\phi}_k dx \right| \leqq \sum_1^n | b_j a_{kj} |$$

$$\leqq \left(\sum_1^n | b_j |^2 \right)^{1/2} \left(\sum_{j=1}^n | a_{kj} |^2 \right)^{1/2}.$$

Hence,

$$\left| b_k - \int_a^b f\bar{\phi}_k dx \right| \leqq \left(\sum_1^n | b_j |^2 \right)^{1/2} \left(\sum_{j=1}^n | a_{kj} |^2 \right)^{1/2} + \left(\int_a^b | f - s_n |^2 dx \right)^{1/2}.$$

Let $n \to \infty$, this becomes

$$\left| b_k - \int_a^b f\bar{\phi}_k dx \right| \leqq \left(\sum_1^\infty | b_j |^2 \right)^{1/2} \left(\sum_{j=1}^\infty | a_{kj} |^2 \right)^{1/2}.$$

Squaring both sides, and summing over k,

$$\sum_k \left| b_k - \int_a^b f\bar{\phi}_k dx \right|^2 \leqq \left(\sum_1^\infty | b_j |^2 \right) \left(\sum_{k,j} | a_{kj} |^2 \right),$$

which is the result in question.

2. An "almost" moment problem. Let us consider the sequence of functions $\{e^{i\lambda_n t}/(b-a)^{1/2}\}$, the λ_n being real and distinct, over a finite interval (a, b). Then we have the following theorem.

THEOREM 3. *If* $\sum_{k \neq l} 1/(\lambda_k - \lambda_l)^2 < \infty$, *and* $\sum_n | b_n |^2 < \infty$, *there exists a function* $f(t) \subset L^2(a, b)$, *such that*

$$\sum_n \left| b_n - \frac{1}{(b-a)^{1/2}} \int_a^b f(t) e^{-i\lambda_n t} dt \right|^2 < \infty.$$

We have

$$\left| \int_a^b e^{i\lambda_l t} e^{-i\lambda_k t} dt \right| \leqq \frac{2}{| \lambda_k - \lambda_l |}, \qquad k \neq l.$$

Therefore, in view of the hypothesis, this is a corollary of Theorem 2.

PRINCETON UNIVERSITY