A RECURRENCE FORMULA FOR THE SOLUTIONS OF CERTAIN LINEAR PARTIAL DIFFERENTIAL EQUATIONS

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1. **Introduction.** In a number of recent papers, Bergman[1] has developed the theory of operational methods for transforming analytic functions of a complex variable into solutions of the linear partial differential equation

$$(1.1)\qquad L(U) = U_{z\bar{z}} + a(z, \bar{z})U_z + b(z, \bar{z})U_{\bar{z}} + c(z, \bar{z})U = 0,$$

where $z=x+iy$, $\bar{z}=x-iy$,

$$U_z = \frac{1}{2}\left(\frac{\partial U}{\partial x} - i\,\frac{\partial U}{\partial y}\right), \qquad U_{\bar{z}} = \frac{1}{2}\left(\frac{\partial U}{\partial x} + i\,\frac{\partial U}{\partial y}\right),$$

$$U_{z\bar{z}} = \frac{1}{4}\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) = \frac{\Delta U}{4},$$

and where the coefficients $a(z, \bar{z})$, $b(z, \bar{z})$ and $c(z, \bar{z})$ are analytic functions of both variables z and $\bar{z}$. The equation (1.1) is equivalent to the system of two real equations

$$\begin{aligned} \Delta U^{(1)} + 2AU_x^{(1)} + 2BU_y^{(1)} + 2CU_x^{(2)} + 2DU_y^{(2)} \qquad & \\ + 4c_1U^{(1)} - 4c_2U^{(2)} &= 0, \\ \Delta U^{(2)} - 2CU_x^{(1)} - 2DU_y^{(1)} + 2AU_x^{(2)} + 2BU_y^{(2)} \qquad & \\ + 4c_2U^{(1)} + 4c_1U^{(2)} &= 0, \end{aligned}$$

where

$$\begin{aligned} &U=U^{(1)}+iU^{(2)}; \quad 2A=(a+\bar{a})+(b+\bar{b}); \quad 2B=i[(\bar{a}-a)-(\bar{b}-b)]; \\ &c=c_1+ic_2; \qquad\quad 2D=(a+\bar{a})-(b+\bar{b}); \quad 2C=i[(a-\bar{a})+(b-\bar{b})]. \end{aligned}$$

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[1] S. Bergman, (a) *Zur Theorie der Funktionen, die eine lineare partielle Differentialgleichung befriedigen*, Rec. Math. (Mat. Sbornik) N.S. vol. 44 (1937) pp. 1169–1198; (b) *The approximation of functions satisfying a linear partial differential equation*, Duke Math. J. vol. 6 (1940) pp. 537–561; (c) *Linear operators in the theory of partial differential equations*, Trans. Amer. Math. Soc. vol. 53 (1943) pp. 130–155; (d) *On the solutions of partial differential equations of the fourth order*, to appear later.

Furthermore, if $a=\bar{b}$ and c is real, then $C=D=c_2=0$, and the two differential equations become real and identical.

For equations (1.1), Bergman proved the existence of two functions $E_1(z, \bar{z}, t)$ and $E_2(z, \bar{z}, t)$, called by him "generating functions of the first kind,"[2] with the following properties:

(1) They have the forms

$$E_1(z, \bar{z}, t) = \exp\left(-\int_0^{\bar{z}} a(z, \bar{z})d\bar{z}\right)[1 + z\bar{z}tE_1^*(z, \bar{z}, t)],$$

$$E_2(z, \bar{z}, t) = \exp\left(-\int_0^{z} b(z, \bar{z})dz\right)[1 + z\bar{z}tE_2^*(z, \bar{z}, t)],$$

where each $E_k^*(z, \bar{z}, t)$ has continuous first partial derivatives in z, $\bar{z}$ and t for $|t| \leq 1$ and for z and $\bar{z}$ within a certain four-dimensional region.

(2) The classes $C(E_1)$ and $C(E_2)$ of functions $U_1(z, \bar{z})$ and $U_2(z, \bar{z})$ defined by the formulas

$$U_1(z, \bar{z}) = \int_{-1}^{1} E_1(z, \bar{z}, t)f(z(1 - t^2)/2)dt/(1 - t^2)^{1/2}, \tag{1.2}$$

$$U_2(z, \bar{z}) = \int_{-1}^{1} E_2(z, \bar{z}, t)g(\bar{z}(1 - t^2)/2)dt/(1 - t^2)^{1/2}, \tag{1.3}$$

where $f(\zeta)$ and $g(\zeta)$ are arbitrary analytic functions of ζ, form subsets of solutions of (1.1).

(3) Every solution $U(z, \bar{z})$ of (1.1) may be written in the form

$$U(z, \bar{z}) = U_1(z, \bar{z}) + U_2(z, \bar{z}),$$

with $f(\zeta)$ and $g(\zeta)$ suitably chosen analytic functions.

As was proved by Bergman, to many theorems about analytic functions of a complex variable correspond analogous theorems about functions belonging to classes $C(E)$ generated by functions E of the first kind. In particular, if we define as "basic solutions" those corresponding to $f(z)=z^p$, that is

$$u_p(z, \bar{z}) = \int_{-1}^{1} E(z, \bar{z}, t)[z(1 - t^2)/2]^p dt/(1 - t^2)^{-1/2}, \tag{1.4}$$

then every function U of class $C(E)$ which is regular in $|z| \leq r$ may

[2] Generating functions which are considered as not of the first kind are those failing to satisfy property (1). When E is a generating function not of the first kind, the integration in (1.2) and (1.3) must be taken along a rectifiable curve joining the points $t=\pm 1$, but not passing through $t=0$.

be expanded in a series $U=\sum\alpha_p u_p$ which is uniformly and absolutely convergent in $|z|\leqq r$.

For example, in the case of the equation

$$\Delta U + U = 0, \tag{1.5}$$

$E(z, \bar{z}, t)=e^{itr}$, where $z=re^{i\theta}$ and thus $r=(z\bar{z})^{1/2}$. Because of the well known formula for the Bessel function of the first kind

$$J_p(r) = (2/\pi)(1/\Gamma(p+1/2))\int_{-1}^{1} e^{rti}(r/2)^p(1-t^2)^{p-1/2}dt,$$

the basic solutions are

$$u_p(r, \theta) = (\pi^{1/2}/2)\Gamma(p+1/2)e^{p\theta i}J_p(r). \tag{1.6}$$

But for (1.5), successive terms in the expansion $U=\sum\alpha_p u_p$ can be computed from earlier terms by the use of some recurrence relation satisfied by the Bessel's functions, as for example the relation

$$J_p'(r) = (p/r)J_p(r) - J_{p+1}(r). \tag{1.7}$$

It would likewise be of practical value in the case of other differential equations $L(U)=0$ to determine what recurrence relations, if any, are satisfied by the basic solutions $u_p(r, \theta)$.

In the present note, recurrence formulas connecting the basic solutions $u_p(r, \theta)$ are found in the case of differential equations $L(U)=0$ for which at least one of the corresponding "generating functions" $E(z, \bar{z}, t)$ is of the form $E(z, \bar{z}, t)=\exp f(r, \theta, t)$ where $f(r, \theta, t)$ is a polynomial in t containing either only even powers of t or only odd powers of t. Obviously, the equation (1.5) is an example of such an equation. Other examples can be found by requiring the coefficients a, b and c in the equation $L(U)=0$ to satisfy certain differential relations.[3]

Our first main result may be stated as follows:

THEOREM 1. *Let $L(U)=0$ be a partial differential equation of the type (1.1) for which there exists a generating function having one of the forms*:

$$E(z, \bar{z}, t) = \exp P(r, \theta, t), \tag{I}$$

$$E(z, \bar{z}, t) = \exp tP(r, \theta, t), \tag{II}$$

where $P(r, \theta, t)=a_0(r, \theta)+a_1(r, \theta)t^2+\cdots+a_n(r, \theta)t^{2n}$, and where the coefficients $a_j(r, \theta)$ are of class C' in r and θ. Let $u_p(r, \theta)$ be the corresponding "basic solutions" of equation $L(U)=0$ and let

[3] See reference in footnote 1a, pp. 1194–1195, and also p. 158 of the following article: K. L. Nielsen and B. P. Ramsay, *On particular solutions of linear partial differential equations*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 156–162.

$$\alpha_q(r,\theta) = (-1)^q\left[\sum_{j=q}^{n} C_{j,q}\frac{\partial a_j}{\partial r}\right]\left(\frac{2}{re^{i\theta}}\right)^q, \tag{1.8}$$

$$\beta_q(r,\theta) = (-1)^q\left[\sum_{j=q}^{n}(2j+1)C_{j,q}a_j\right]\left(\frac{2}{re^{i\theta}}\right)^q. \tag{1.9}$$

Then, if E has form (I),

$$\frac{\partial u_p}{\partial r} = \frac{p}{r}u_p + \sum_{j=0}^{n}\alpha_j u_{p+j}; \tag{1.10}$$

whereas, if E has form (II),

$$\frac{\partial u_p}{\partial r} = \frac{p}{r}u_p + \frac{2}{re^{i\theta}}\sum_{j,k=0}^{n}\frac{\alpha_j\beta_k u_{j+k+p+1}}{2j+2p+1}. \tag{1.11}$$

The above theorem will be derived as an immediate consequence of two lemmas that are given in the next section. In the third section the theorem will be applied to a few specific equations of form (1.1).

2. **Two lemmas.** First we shall derive a result for polynomials $P(r, \theta, t)$ involving only even powers of t.

LEMMA 1. *Let*

$$P(r,\theta,t) = a_0(r,\theta) + a_1(r,\theta)t^2 + \cdots + a_n(r,\theta)t^{2n},$$

where the $a_j(r, \theta)$ are functions of class C' in r and θ. Then the function

$$u_p(r,\theta) = \int_{-1}^{1} e^{P(r,\theta,t)}(1-t^2)^{p-1/2}(re^{i\theta}/2)^p dt \tag{2.1}$$

satisfies the recurrence formula

$$\partial u_p/\partial r = [p/r + P_r(r,\theta,(1-T)^{1/2})]u_p, \tag{2.2}$$

where T is the operator such that, T acting k times upon u_p,

$$T^k u_p = (2/re^{i\theta})^k u_{p+k} \tag{2.3}$$

for $k=0, 1, \cdots$.

PROOF. From (2.1) we obtain by differentiating with respect to r:

$$\begin{aligned}\frac{\partial u_p}{\partial r} = \frac{p}{r}u_p + \int_{-1}^{1} e^P\Big[\frac{\partial a_0}{\partial r} + \frac{\partial a_1}{\partial r}t^2 + \cdots \\ + \frac{\partial a_n}{\partial r}t^{2n}\Big](1-t^2)^{p-1/2}\left(\frac{re^{i\theta}}{2}\right)dt.\end{aligned} \tag{2.4}$$

To evaluate the latter integral, let us note that

$$\begin{aligned}\int_{-1}^{1} & t^{2m}e^{P}(1-t^2)^{p-1/2}(re^{i\theta}/2)^p dt \\ &= \int_{-1}^{1}[1-(1-t^2)]^m e^P(1-t^2)^{p-1/2}(re^{i\theta}/2)^p dt \\ &= \sum_{k=0}^{m}(-1)^k C_{m,k}\int_{-1}^{1} e^P(1-t^2)^{k+p-1/2}(re^{i\theta}/2)^p dt \\ &= \sum_{k=0}^{m}(-1)^k C_{m,k}(2/re^{i\theta})^p u_{p+k}.\end{aligned}$$

Hence,

$$\int_{-1}^{1} t^{2m}e^P(1-t^2)^{p-1/2}(re^{i\theta}/2)^p dt = (1-T)^m u_p. \tag{2.5}$$

Substituting now from (2.5) into (2.4), we find

$$\begin{aligned}\frac{\partial u_p}{\partial r} &= \frac{p}{r}u_p + \frac{\partial a_0}{\partial r}u_p + \frac{\partial a_1}{\partial r}(1-T)u_p + \cdots + \frac{\partial a_n}{\partial r}(1-T)^n u_p \\ &= [p/r + P_r(r, \theta, (1-T)^{1/2})]u_p,\end{aligned}$$

as was to be proved.

The corresponding result for a polynomial that involves only odd powers of t may be stated as follows.

LEMMA 2. *Let* $Q(r, \theta, t) \equiv a_0(r, \theta)t + a_1(r, \theta)t^3 + \cdots + a_n(r, \theta)t^{2n+1} \equiv tP(r, \theta, t)$, *where the* $a_j(r, \theta)$ *are functians of class* C' *in* r *and* θ. *Then the functions* $u_p(r, \theta)$ *defined by* (2.1) *satisfy the recurrence formula*:

$$\begin{aligned}\partial u_p/\partial r = (p/r)u_p + \{Q_t(r, \theta, (1-T)^{1/2}) \\ \cdot \int_0^{T^{1/2}} t^{2p}P_r(r, \theta, (1-t)^{1/2})dt\} T^{-p+1/2}u_p ,\end{aligned} \tag{2.6}$$

where T *is the operator defined by equation* (2.3).

PROOF. In place of (2.4), we now have

$$\begin{aligned}\frac{\partial u_p}{\partial r} = \frac{p}{r}u_p + \int_{-1}^{1} e^Q\Big[\frac{\partial a_0}{\partial r}t + \frac{\partial a_1}{\partial r}t^3 + \cdots \\ + \frac{\partial a_n}{\partial r}t^{2n+1}\Big](1-t^2)^{p-1/2}\left(\frac{re^{i\theta}}{2}\right)^p dt.\end{aligned} \tag{2.7}$$

In order to evaluate the latter integral, let us first integrate by parts:

$$\int_{-1}^{1} te^{Q}(1-t^2)^{p-1/2}(re^{i\theta}/2)^p dt$$
$$= -\frac{1}{2}\int_{-1}^{1} e^{Q}(1-t^2)^{p-1/2}(re^{i\theta}/2)^p d(1-t^2)$$
$$= (1/(2p+1))\int_{-1}^{1} e^{Q}(\partial Q/\partial t)(1-t^2)^{p+1/2}(re^{i\theta}/2)^p dt$$
$$= (1/(2p+1))\int_{-1}^{1} e^{Q}\big[a_0 + 3a_1t^2 + \cdots$$
$$+ (2n+1)a_nt^{2n}\big](1-t^2)^{p+1/2}(re^{i\theta}/2)^p dt.$$

Hence, by equation (2.5),

$$\int_{-1}^{1} te^{Q}(1-t^2)^{p-1/2}(re^{i\theta}/2)^p dt = (1/(2p+1))Q_t(r,\theta,(1-T)^{1/2})Tu_p$$
$$= Q_t(r,\theta,(1-T)^{1/2})\left(\int_0^{T^{1/2}} t^{2p}dt\right)T^{-p+1/2}u_p.$$

Let us then assume that the formula

$$\int_{-1}^{1} t^{2m+1}e^{Q}(1-t^2)^{p-1/2}(re^{i\theta}/2)^p dt$$
$$= Q_t(r,\theta,(1-T)^{1/2})\left(\int_0^{T^{1/2}} t^{2p}(1-t^2)^m dt\right)T^{-p+1/2}u^p \tag{2.8}$$

has already been verified for $m=0, 1, 2, \cdots, N$ and proceed to verify the formula for $m=N+1$, as follows.

$$\int_{-1}^{1} t^{2N+3}e^{Q}(1-t^2)^{p-1/2}(re^{i\theta}/2)^p dt$$
$$= \int_{-1}^{1} t^{2N+1}e^{Q}(1-t^2)^{p-1/2}(re^{i\theta}/2)^p dt$$
$$- \int_{-1}^{1} t^{2N+1}e^{Q}(1-t^2)^{p+1/2}(re^{i\theta}/2)^p dt$$
$$= Q_t(r,\theta,(1-T)^{1/2})\left\{\left[\int_0^{T^{1/2}} t^{2p}(1-t^2)^N dt\right]T^{-p+1/2}u_p\right.$$
$$\left.- \left[\int_0^{T^{1/2}} t^{2p+2}(1-t^2)^N dt\right](2/re^{i\theta})T^{-p-1/2}u_{p+1}\right\}$$
$$= Q_t(r,\theta,(1-T)^{1/2})\left(\int_0^{T^{1/2}} t^{2p}(1-t^2)^{N+1}dt\right)T^{-p+1/2}u_p.$$

Thus, formula (2.8) has been established by mathematical induction.

We now substitute from formula (2.8) into expression (2.7), thus obtaining

$$
\begin{aligned}
\frac{\partial u_p}{\partial r} &= \frac{p}{r}\, u_p + \Bigg(Q_t(r,\theta,(1-T)^{1/2}) \\
&\qquad \cdot \int_0^{T^{1/2}} t^{2p}\left\{ \frac{\partial a_0}{\partial r} + \frac{\partial a_1}{\partial r}(1-t^2) + \cdots + \frac{\partial a_n}{\partial r}(1-t^2)^n \right\} \Bigg) T^{-p+1/2} u_p \\
&= \frac{p}{r}\, u_p + Q_t(r,\theta,(1-T)^{1/2}) \left\{ \int_0^{T^{1/2}} t^{2p} P_r(r,\theta,(1-t)^{1/2}) \right\} T^{-p+1/2} u_p,
\end{aligned}
$$

as was to be proved.

PROOF OF THEOREM 1. Formula (2.2) may be reduced to formula (1.10) if, using (1.8) and (2.3), we set

$$
\begin{aligned}
P_r(r,\theta,(1-T)^{1/2})u_p &= \sum_{j=0}^{n} \frac{\partial a_j}{\partial r}(1-T)^j u_p = \sum_{q=0}^{n} \alpha_q \left(\frac{re^{i\theta}}{2}\right)^q T^q u_p \\
&= \sum_{q=0}^{n} \alpha_q u_{p+q}.
\end{aligned}
$$

To reduce formula (2.6) to (1.11), let us set

$$
Q_t(r,\theta,(1-T)^{1/2}) = \sum_{j=0}^{n} (2j+1) a_j (1-T)^j = \sum_{j=0}^{n} \beta_j \left(\frac{re^{i\theta}}{2}\right)^j T^j
$$

with the β_j defined as in formula (1.9). Since

$$
P_r(r,\theta,(1-t^2)^{1/2}) = \sum_{j=0}^{n} \alpha_j \left(\frac{re^{i\theta}}{2}\right)^j t^{2j},
$$

we may write the second term of the left side of (2.6) as

$$
\begin{aligned}
&\sum_{k=0}^{n} \beta_k \left(\frac{re^{i\theta}}{2}\right)^k T^k \left[\int_0^{T^{1/2}} t^{2p} \sum_{j=0}^{n} \alpha_j \left(\frac{re^{i\theta}}{2}\right)^j t^{2j} dt \right] T^{-p+1/2} u_p \\
&\quad = \sum_{j,k=0}^{n} \frac{\alpha_j \beta_k}{2p+2j+1} \left(\frac{re^{i\theta}}{2}\right)^{j+k} T^{k+j+1} u_p = \frac{2}{re^{i\theta}} \sum_{j,k=0}^{n} \frac{\alpha_j \beta_k u_{p+j+k+1}}{2p+2j+1}.
\end{aligned}
$$

Thus the proof of our main theorem is completed.

3. **Examples.** Let us first verify that the recurrence relation (2.6) is a generalization of that for Bessel's functions as given in formula (1.7). Here $Q(r, \theta, t) = rti$ and thus (2.6) becomes

$$\begin{aligned}\partial u_p/\partial r &= (p/r)u_p + ri\left[\int_0^{T^{1/2}} t^{2p}idt\right]T^{-p+1/2}u_p \\ &= (p/r)u_p - (2e^{-i\theta}/(2p+1))u_{p+1}.\end{aligned} \tag{3.1}$$

If now we set

$$\begin{aligned} u_p &= (\pi^{1/2}/2)\Gamma(p+1/2)J_p(r)e^{ip\theta}, \\ u_{p+1} &= (\pi^{1/2}/2)(p+1/2)\Gamma(p+1/2)J_{p+1}(r)e^{i(p+1)\theta}, \end{aligned}$$

formula (3.1) reduces at once to formula (1.7).

As our second example, let us consider the differential equation $L(U)=0$ in which the expression $F=c-ab-a_z\neq 0$ satisfies the two equations

$$F_z = 0, \qquad 2F - a_z + b_{\bar{z}} = 0. \tag{3.2}$$

As shown by Bergman,[4] one of the possible corresponding generating functions is $E(z, \bar{z}, t) = \exp P(r, \theta, t) = \exp(a_0 + a_1 t^2)$, where

$$a_0 = -\int_0^{\bar{z}} a d\bar{z}, \qquad a_1 = 2z\int_0^{\bar{z}} F d\bar{z}. \tag{3.3}$$

According to our theorem, the recurrence relation satisfied by the basic solutions is in this case

$$(\partial u_p/\partial r) = (p/r)u_p + \alpha_0 u_p + \alpha_1 u_{p+1}, \tag{3.4}$$

where

$$\begin{aligned} \alpha_0 &= \partial a_0/\partial r + \partial a_1/\partial r = -ae^{-i\theta} + 2rF + 2e^{i\theta}\int_0^{\bar{z}} F d\bar{z}, \\ \alpha_1 &= -(2/re^{i\theta})(\partial a_1/\partial r) = -4e^{-i\theta}F - (4/r)\int_0^{\bar{z}} F d\bar{z}. \end{aligned}$$

A partial differential equation which satisfies conditions (3.2) is

$$U_{z\bar{z}} - 2(z+\bar{z})U_{\bar{z}} + U = 0. \tag{3.5}$$

Here $F(z)=1$, $a_0=0$, $a_1=2r^2$, and therefore in the recurrence relation (3.4) $\alpha_0=4r$, and $\alpha_1=-8e^{-i\theta}$. Setting $U=U^{(1)}+iU^{(2)}$, we see that equation (3.5) is equivalent to the system of two partial differential equations

$$\begin{aligned} \Delta U^{(1)} - 8xU_x^{(1)} + 8xU_y^{(2)} + 4U^{(1)} &= 0, \\ \Delta U^{(2)} - 8xU_x^{(2)} - 8xU_y^{(1)} + 4U^{(2)} &= 0, \end{aligned}$$

[4] See p. 1194, reference in footnote 1a.

and that recurrence relation (3.4) for $u_p = u_p^{(1)} + iu_p^{(2)}$ is in this case equivalent to the system of recurrence relations

$$\partial u_p^{(1)}/\partial r = (p/r)u_p^{(1)} + 4ru_p^{(1)} - 8u_{p+1}^{(1)}\cos\theta - 8u_{p+1}^{(2)}\sin\theta,$$
$$\partial u_p^{(2)}/\partial r = (p/r)u_p^{(2)} + 4ru_p^{(2)} - 8u_{p+1}^{(2)}\cos\theta + 8u_{p+1}^{(1)}\sin\theta.$$

Other examples of partial differential equations $L(U)=0$ for which $\log E$ is an even or odd polynomial in t may be found in the articles referred to in footnotes 1a and 3. For these differential equations also, a recurrence relation may be derived by use of Lemmas 1 and 2.

4. **Generalization.** By means of formulas (2.5) and (2.8), the theorem given in the introduction may be extended to partial differential equations of type (1.1) for which a generating function exists that has the form $E = g \exp f$ with both f and g suitably chosen polynomials in t. The generalization may be stated as follows.

THEOREM 2. *Let $L(U)=0$ be a partial differential equation of type (1.1) for which a generating function $E(z, \bar{z}, t)$ exists that has one of the forms*

$$\text{I.}\quad E(z, \bar{z}, t) = R(r, \theta, t)\exp P(r, \theta, t),$$
$$\text{II.}\quad E(z, \bar{z}, t) = R(r, \theta, t)\exp tP(r, \theta, t),$$
$$\text{III.}\quad E(z, \bar{z}, t) = tR(r, \theta, t)\exp tP(r, \theta, t),$$

where

$$P(r, \theta, t) = a_0(r, \theta) + a_1(r, \theta)t^2 + \cdots + a_m(r, \theta)t^{2m},$$
$$R(r, \theta, t) = b_0(r, \theta) + b_1(r, \theta)t^2 + \cdots + b_n(r, \theta)t^{2n},$$

and where the $a_j(r, \theta)$ and $b_j(r, \theta)$ are of class C' in r and θ. Let $u_p(r, \theta)$ be the corresponding basic solutions and let $R(\partial p/\partial r) = \sum_0^{m+n} c_j(r, \theta)t^{2j}$,

$$\alpha_k = (-1)^k\left(\frac{2}{re^{i\theta}}\right)^k \sum_{j=k}^{m}(2j+1)C_{j,k}a_j;$$

$$\beta_k = (-1)^k\left(\frac{2}{re^{i\theta}}\right)^k \sum_{j=k}^{n} C_{j,k}\frac{\partial b_j}{\partial r};\qquad \gamma_k = (-1)^k\left(\frac{2}{re^{i\theta}}\right)^k \sum_{j=k}^{n+m} C_{j,k}c_j.$$

Then the recurrence relation satisfied by these basic solutions is

$$\frac{\partial u_p}{\partial r} = \frac{pu_p}{r} + \sum_{k=0}^{n}\beta_k u_{p+k} + \sum_{k=0}^{n+m}\gamma_k u_{p+k}$$

if E has the form I;

$$\frac{\partial u_p}{\partial r} = \frac{pu_p}{r} + \sum_{k=0}^{n} \beta_k u_{p+k} + \left(\frac{2}{re^{i\theta}}\right) \sum_{k=0}^{m} \sum_{\nu=0}^{m+n} \frac{\alpha_k \gamma_\nu u_{p+k+\nu+1}}{2p + 2\nu + 1}$$

if E has the form II; *and*

$$\frac{\partial u_p}{\partial r} = \frac{pu_p}{r} + \left(\frac{2}{re^{i\theta}}\right) \sum_{k=0}^{m} \sum_{\nu=0}^{n} \left(\frac{\alpha_k \beta_\nu}{2p + 2\nu + 1}\right) u_{p+k+\nu+1}$$
$$+ \gamma_0 u_p + \sum_{k=1}^{m+n} \left(\gamma_k - \frac{2}{re^{i\theta}} \gamma_{k-1}\right) u_{p+k}$$

if E has the form III.

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ON GAUSS' AND TCHEBYCHEFF'S QUADRATURE FORMULAS

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The well known Gauss' Quadrature Formula

$$\int_{-\infty}^{\infty} G_k(x) d\psi(x) = \sum_{i=1}^{n} \rho_i^{(n)} G_k(\xi_i^{(n)}) \tag{1}$$

is valid for every polynomial $G_k(x)$, of degree $k \leqq 2n-1$, the $\{\xi_i^{(n)}\}$ being the roots of the polynomial $P_n(x)$, orthogonal with respect to the distribution $d\psi(x)$ $(i=1, 2, \cdots, n;\ n=1, 2, \cdots)$.[1] If the sequence $\{P_n(x)\}$ is that of Tchebycheff (trigonometric) polynomials, then the Christoffel numbers $\rho_i^{(n)}$, $i=1, 2, \cdots, n$, are equal, and the two quadrature formulas of Gauss and Tchebycheff coincide:

$$\int_{-\infty}^{\infty} G_k(x) d\psi(x) = \rho_n \sum_{i=1}^{n} G_k(\xi_i^{(n)}), \qquad k \leqq 2n - 1;\ n = 1, 2, \cdots. \tag{2}$$

The converse—that this is the only case of coincidence of these formulas—was proved by R. P. Bailey [1a] and, under more restrictive conditions, by Krawtchouk [1b] (cf. also [2]).[2]

We shall give here four distinct proofs of this statement, without imposing any restrictions on $\psi(x)$.

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[1] $\psi(x)$ is a bounded non-decreasing function, with infinitely many points of increase, for which all moments exist: $c_n = \int_{-\infty}^{\infty} x^n d\psi(x)$; $n=0, 1, 2, \cdots$.

[2] Numbers in brackets refer to the bibliography at the end of the paper.