MODULARITY IN BIRKHOFF LATTICES

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The purpose of this note is to identify upper semi-modular lattices originally defined by G. Birkhoff[1] and subsequently studied by Dilworth[2] with those M-symmetric lattices[3] (introduced independently by the author without assumption of chain conditions) which satisfy a condition of finite dimensionality.

The definitions and notations are these. In a lattice L, $a \succ b (b \prec a)$ means that a "covers" b, that is, $a > b$, together with $a \geqq x \geqq b$ implies $x = a$ or $x = b$; $(b, c)M$ means $(a+b)c = a+bc$ for every $a \leqq c$ (where $a+b$, ab are the "join" and "meet" respectively of a, b). We say that L is *M-symmetric* if the binary relation M is symmetric; L is a *Birkhoff lattice* if

$$(1) \qquad a, b \succ ab \textit{ implies } a + b \succ a, b;$$

L is *of finite-dimensional type*[4] if for every $a < b$ there exists a finite "principal chain"

$$a_1 \prec a_2 \prec \cdots \prec a_n,$$

with $a_1 = a$, $a_n = b$. When a, b satisfy this condition for a specific n, we say that *b is $n-1$ steps over a*.

The properties of the relation M are given in part in a previous paper.[5] Additional properties needed here are contained in the following lemma.

LEMMA 1. *Suppose $b, c \in L$. Then*
(a) *$(b, c)M$ if and only if $bc \leqq a \leqq c$ implies $(a+b)c = a$;*
(b) *if $(b, c)M$, then $(b', c')M$ for $bc \leqq b' \leqq b$, $bc \leqq c' \leqq c$.*

PROOF. The forward implication in (a) is obvious. To prove the

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[1] G. Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloquium Publications vol. 25, New York, 1940, p. 62.

[2] R. P. Dilworth, *Ideals in Birkhoff lattices*, Trans. Amer. Math. Soc. vol. 49 pp. 325–353; also *The arithmetical theory of Birkhoff lattices*, Duke Math. J. vol. 8 (1941) pp. 286–299.

[3] L. R. Wilcox, *Modularity in the theory of lattices*, Ann. of Math. vol. 40 (1939) pp. 490–505; see also *A note on complementation in lattices*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 453–457.

[4] This property is weaker than finite dimensionality as used by Birkhoff (loc. cit. p. 11), even if 0 and 1 exist.

[5] L. R. Wilcox, *Modularity in the theory of lattices*, pp. 491–495.

converse, let $a \leqq c$. Then $a' = a + bc$ has the property $bc \leqq a' \leqq c$, whence

$$(a + b)c = (a + bc + b)c = (a' + b)c$$
$$= a' = a + bc.$$

To prove (b) we use the condition in (a). Let $b'c' \leqq a \leqq c'$. Then

$$(a + b')c' \leqq (a + b)cc' = ac' = a \leqq (a + b')c',$$

whence (b) follows.

THEOREM 1. *Every M-symmetric lattice is a Birkhoff lattice.*

PROOF. Suppose $a, b \succ ab$. Then it is immediate that $a+b > a, b$. To prove $a+b \succ a$, let $a \leqq c \leqq a+b$. Since $b \geqq cb \geqq ab$, we have $cb = b$ or $cb = ab$ from the hypothesis $b \succ ab$. If $cb = b$, then $a+b \leqq c$, whence $c = a+b$. Suppose $cb = ab$. We shall prove $(c, b)M$. Let $ab = cb \leqq x \leqq b$. Then $x = ab$ or $x = b$, whence either

$$(x + c)b = (ab + c)b = (cb + c)b = cb = x,$$

or

$$(x + c)b = (b + c)b = b = x,$$

and it follows by Lemma 1 (a) that $(c, b)M$. Now the symmetry of M yields $(b, c)M$, and thus, since $bc \leqq a \leqq c$,

$$c = (a + b)c = a.$$

In all cases $c = a+b$ or $c = a$, and consequently $a+b \succ a$. Similarly $a+b \succ b$.

REMARK. The theorem just proved generalizes the known result[6] that every modular lattice is a Birkhoff lattice, since modular lattices are M-symmetric.

In order to consider the converse of Theorem 1, let, for the purposes of the following lemmas, L be a fixed Birkhoff lattice of finite-dimensional type.

LEMMA 2. *If $b, c \in L$ and $c \succ bc$, then $b+c \succ b$.*

PROOF.[7] Observe that $b \geqq bc$; if $b = bc$, $b \leqq c$, and $b+c = c \succ bc = b$. If $b > bc$, then there exists $n = 1, 2, \cdots$ such that b is n steps over bc. If $n = 1$, the result is obvious from condition (1) defining a Birkhoff lattice. Suppose the result has been proved for all b, c for which b is

[6] Birkhoff, loc. cit. p. 34.

[7] This is MacLane's second "exchange axiom" in the convex lattice of all $x \geqq bc$; as such it follows for finite-dimensional lattices from remarks on p. 63 of Birkhoff. Since L need not be finite-dimensional, we give the proof in full.

k steps over bc, and let b be $k+1$ steps over bc. Clearly there exists $b' \prec b$ such that b' is k steps over bc. Since $b'c \leqq bc \leqq b'c$, we have $c \succ b'c$, and by the induction hypothesis applied to b', c it follows that $b'+c \succ b'$. But $b' \leqq (b'+c)b \leqq b$, whence $(b'+c)b = b'$ or $(b'+c)b = b$. In the latter case $b' \prec b \leqq b'+c$, and thus $b = b'+c$, whence $c \leqq b$, contrary to $c \succ bc$. Consequently $(b'+c)b = b'$. Since $b'+c$, $b \succ (b'+c)b$, (1) yields

$$b + c = b + (b' + c) \succ b.$$

LEMMA 3. *If b, $c \in L$, $c \succ bc$, then $(c, b)M$, $(b, c)M$.*

PROOF. If $bc \leqq a \leqq c$, then $a = bc$ or $a = c$, so that either

$$(a + b)c = (bc + b)c = bc = a,$$

or

$$(a + b) c = (c + b)c = c = a,$$

and $(b, c)M$. Now suppose $bc \leqq a \leqq b$. Then $bc \leqq ac \leqq bc$ yields $ac = bc$. Hence $c \succ ac$, and $a+c \succ a$ by Lemma 2. But $a \leqq (a+c)b \leqq a+c$, whence $(a+c)b = a$ or $(a+c)b = a+c$. In the latter case $a+c \leqq b$, and $c \leqq b$, which is impossible. Hence $(a+c)b = a$, and $(c, b)M$.

LEMMA 4. *Suppose b, $c \in L$, $(b, c)M$. Then $bc \leqq a \leqq b$, $a+c = b+c$ implies $a = b$.*

PROOF. If $c = bc$, that is, $c \leqq b$, or if c is one step over bc then $(c, b)M$ either by direct verification or by Lemma 3; hence

$$a = a + cb = (a + c)b = (b + c)b = b.$$

Suppose the result holds for all b, c with c n steps over bc, and let b, c satisfy the hypotheses, c being $n+1$ steps over bc. Then there exists c' with $bc \leqq c' \prec c$, where c' is n steps over bc. Since $(b, c')M$ by Lemma 1 (b), and since $bc' = bc \leqq a \leqq b$, we need only verify $a+c' = b+c'$ in order to show $a = b$. Since $(a, c)M$ by Lemma 1 (b), $(c'+a)c = c'$. Thus

$$c \succ c' = (c' + a)c,$$

and by Lemma 2,

$$c + b = c + a = c + (c' + a) \succ c' + a.$$

But

$$c' + a \leqq c' + b \leqq c + b,$$

whence

$$c' + b = c' + a \quad \text{or} \quad c' + b = c + b.$$

In the second case, since $(b, c)M$,

$$c' = (c' + b)c = (c + b)c = c,$$

which is impossible. This completes the proof.

THEOREM 2. *Every Birkhoff lattice L of finite-dimensional type is M-symmetric.*

PROOF. Suppose $(b, c)M$, and in proof of $(c, b)M$ let $bc \leqq a \leqq b$. Define

$$b_1 = (a + c)b \geqq a;$$

we shall prove that $b_1 = a$ by applying Lemma 4 to a, b_1, c in place of a, b, c. First, $(b_1, c)M$ by Lemma 1 (b), since $bc \leqq b_1 \leqq b$, and $(b, c)M$. Moreover,

$$b_1c = (a + c)cb = bc \leqq a \leqq b_1.$$

Finally, $a + c \geqq b_1$, c, whence

$$a + c \geqq b_1 + c \geqq a + c,$$

and $a + c = b_1 + c$. The hypotheses of Lemma 4 have been verified, and thus $a = b_1$, as was to be proved.

The effect of Theorems 1 and 2 is to show that not necessarily finite-dimensional M-symmetric lattices are a true generalization of the Birkhoff lattices. Moreover, the condition defining M-symmetry does not lose its strength in infinite-dimensional cases as does condition (1). For example, an interval of real numbers ordered as usual satisfies (1) vacuously; it is modular, hence M-symmetric. However, define a lattice L as consisting of the closed real interval $I = [0, 1]$, ordered naturally, together with an element ϵ, with $0 < \epsilon < 1$, but $x \nless \epsilon$, $\epsilon \nless x$, $\epsilon \neq x$ for $x \in I$. This is a lattice in which the only covering relations are $\epsilon \succ 0$, $1 \succ \epsilon$. Hence (1) is vacuously true, but M-symmetry fails violently, since $(x, \epsilon)M$ for every $x \in L$, but $(\epsilon, x)M$ is false except for $x = 0$, 1 or ϵ.

Interesting questions are these. What infinite-dimensional generalization of the Jordan chain condition holds in M-symmetric lattices? Moreover, in finite-dimensional lattices, (1) together with its dual implies modularity; what can be said generally of lattices which together with their duals are M-symmetric?

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