ON r-REGULAR CONVERGENCE

PAUL A. WHITE

In his paper *On sequences and limiting sets* [1],[1] G. T. Whyburn introduced the notion of regular convergence. He showed that in the cases of 0 and 1 regular convergence (see definition below) that the limit of sequences of many simple topological sets is of the same type as the members of the sequence. It is the purpose of this paper to extend some of these results to higher dimensions. The lack of simple characterizations of the higher dimension sets (such as the n-sphere) makes the results much weaker than in the 0 and 1 dimensional cases.

It is assumed throughout the paper that all sets lie in a compact metric space. All our complexes and cycles will be non-oriented, and the Vietoris cycles and chains (V-cycles and V-chains) will have these as coordinates. The set of all points x whose distance from a set A is less than ϵ will be denoted by $U_\epsilon(A)$. Finally we shall denote the boundary of an r-dimensional complex (or V-chain) z^r by $\dot{z}^r$.

DEFINITION. *A sequence of closed sets* (A_i) *converging to a limit set* A *is said to converge r-regularly* $(\rightarrow r)$ *if for every* $\epsilon > 0$ *there exist numbers* $\delta > 0$ *and* $N > 0$ *such that, if* $n > N$, *any r-dimensional V-cycle in* A_n *of diameter less than* δ *is* ~ 0 *in a subset of* A_n *of diameter less than* ϵ. *If* $A_i \rightarrow sA$ *for all* $s \leqq r$, *we write* $A_i \rightarrow \leqq rA$ [1].

DEFINITION. *A Vietoris cycle* $\xi^r = (x_i^r)$ *is called a projection cycle if* $\lim_{i\to\infty}$ *(point set* $x_i^r) = X$ *and each* $x_i^r \subset X$. *Clearly* X *is the smallest carrier* [2] *of* ξ^r.

Note. Corresponding to any cycle $\xi^r = (x_i^r)$ of a compact set F, there always exists a projection cycle $\xi_1^r \sim \xi^r$ in F. In fact if a convergent subsequence of (x_i^r) is chosen, this set can be used as the set X of the definition.

THEOREM 1. *If* $A_i \rightarrow rA$, *then for any* $\epsilon > 0$ *there exist positive numbers* δ *and* N *such that if* x^{r+1} *is a simplex of* A_i $(i > N)$ *whose boundary has a V-chain realization* [3] *of diameter less than* δ, *then* x^{r+1} *has a V-chain realization of diameter less than* ϵ.

PROOF. Let δ and N be the numbers corresponding to ϵ in the definition of r-regular convergence and consider a simplex x^{r+1} of A_i $(i > N)$

Presented to the Society, September 13, 1943; received by the editors September 30, 1943.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

whose boundary has a V-chain realization $\eta^r=(y_i^r)$ of diameter less than δ. By the choice of δ and N, $\eta^r\sim 0$ in a subset of A_i of diameter less than ϵ. Thus there is a V-chain $\xi^{r+1}=(x_i^{r+1})$ of diameter less than ϵ such that $\dot{x}_i^{r+1}=y_i^r$. Clearly ξ^{r+1} is the V-chain realization of x^{r+1} of diameter less than ϵ.

THEOREM 2. *If $A_i\to\leqq rA$ then for any $\epsilon>0$ there exist positive numbers $\delta^r(\epsilon)$ and $N^r(\epsilon)$ such that if x^s $(s\leqq r+1)$ is a $\delta^r(\epsilon)$-simplex of A_i $(i\geqq N^r(\epsilon))$, then x^s has a V-chain realization in a subset of A_i of diameter less than ϵ.*

PROOF. The proof shall be by induction.

The case $r=0$ is clearly a direct consequence of the definition of 0-regular convergence where the numbers $\delta^0(\epsilon)$ and $N^0(\epsilon)$ are the numbers δ and N of the definition.

Suppose the theorem to be true for $r=k-1$ and consider the case $r=k$. Let ϵ be an arbitrary positive number and $\delta<\epsilon$, N be the numbers corresponding to it in Theorem 1. Now by hypothesis there exist positive numbers $\delta^{k-1}(\delta/3)$ and $N^{k-1}(\delta/3)$, and we define $\delta^k(\epsilon)=\min(\delta^{k-1}(\delta/3),\ \delta/3)$ and $N^k(\epsilon)=N^{k-1}(\delta/3)+N$. Let x^s $(s\leqq k+1)$ be a $\delta^k(\epsilon)$-simplex of A_i $(i>N^k(\epsilon))$. If $s\leqq k$, we know that x^s has a V-chain realization of diameter less than $\delta/3<\epsilon$ since $\delta^k(\epsilon)\leqq\delta^{k-1}(\delta/3)$ and $N^k(\epsilon)>N^{k-1}(\delta/3)$. If $s=k+1$, the boundary of x^{k+1} is a k-dimensional $\delta^{k-1}(\delta/3)$-cycle of A_i $(i>N^{k-1}(\delta/3))$. Thus each simplex of this cycle has a V-chain realization in a subset of A_i of diameter less than $\delta/3$. Furthermore we may suppose these realizations to be chosen so that common sides of two simplices have the same realizations. Now adding these V-chains for the simplices of the cycle we obtain a sequence of k-dimensional cycles, a subsequence of which yields a k-dimensional V-cycle of diameter less than $\delta^k(\epsilon)+2(\delta/3)<\delta/3+2(\delta/3)=\delta$ which is clearly a realization of the boundary of x^{k+1}. Now by the choice of δ, we know that x^{k+1} has a V-chain realization in a subset of A_i $(i>N^k(\epsilon)>N)$ of diameter less than ϵ. Thus the theorem is true for $r=k$ and hence for all cases.

THEOREM 3. *If $A_i\to\leqq(r-1)A$ and C is the smallest carrier of an essential projection cycle $\xi^r=(x_j^r)$ of A, then C can be expressed as $\lim_{i\to\infty}C_i$ where C_i is the smallest carrier of a cycle ξ_i^r of A_i which will be essential for all sufficiently large i.*

PROOF. Since ξ^r is essential, there is a positive number η such that ξ^r is not $\sim_\eta 0$ in C. Let $\epsilon_k\to 0$ be a sequence of positive numbers and let $3\delta_k=\min\ \delta^{r-1}(\epsilon_k/3),\ \delta^{r-1}(\eta/3)$ (from Theorem 2). Choose N_k' such that for $i>N_k'$ we have $U_{\delta_k}(A_i)\supset A$, $U_{\delta_k}(A)\supset A_i$ and let $N_k''=N_k'$

$+N^{r-1}(\delta_k/3)+N^{r-1}(\eta/3)$. This defines a number N_k'' for each k, and by letting $N_k=\sum_{j=1}^{k}N_j''$ we obtain a monotone increasing sequence of numbers. Consider a δ_k-cycle $x_{j(k)}^r$ of ξ^r that is not $\sim_\eta 0$ in C and such that $U_{\delta_k}(x_{j(k)}^r)\supset C$. Now pick a number $n>N_k$ and let $a_0, a_1, \cdots, a_g$ be the vertices of $x_{j(k)}^r$, then for each $s\leqq g$, let b_s be a point of A_n such that $\rho(a_s, b_s)<\delta_k$. For each simplex $(a_{i_0}, a_{i_1}, \cdots, a_{i_r})$ in $x_{j(k)}^r$ let $(b_{i_0}, b_{i_1}, \cdots, b_{i_r})$ be a simplex and let x^r be the cycle consisting of these simplices. Clearly x^r will be a $3\delta_k$-cycle of A_n, and we shall call any cycle obtained in this manner a δ_k-projection of $x_{j(k)}^r$ [1]. Our choice of $3\delta_k$ allows us to realize each simplex of x^r in a subset of A_n of diameter less than $\epsilon_k/3$, $\eta/3$, from which we obtain a V-cycle realization $\xi_n^r=(x_{jn}^r)$ of x^r in $U_{\epsilon_k/3}(x^r)$ as well as in $U_{\eta/3}(x^r)$. Now C_n, the smallest carrier of ξ_n^r, will satisfy the conditions of our theorem, for $C_n\subset\overline{U}_{\epsilon_k/3}(x^r)$, $x^r\subset U_{\delta_k}(x_{j(k)}^r)$, $x_{j(k)}^r\subset U_{\delta_k}(x^r)$, $C\subset U_{\delta_k}(x_{j(k)}^r)$. Therefore, $C_n\subset U_{\epsilon_k}(C)$, $C\subset U_{\epsilon_k}(C_n)$ as $(\epsilon_k/3)+\delta_k+\delta_k<\epsilon_k/3+\epsilon_k/3+\epsilon_k/3=\epsilon_k$. Now for all n such that $N_k\leqq n<N_{k+1}$, choose C_n corresponding to ϵ_k; then $\lim_{n\to\infty} C_n=C$.

Finally, since $\xi_n^r\subset U_{\eta/3}(x^r)$, we obtain by an application of the prism construction [4] the homology $x_{jn}^r\sim_{\eta/3}x^r$ for all j. Thus ξ_n^r is not $\sim_{\eta/3}0$ in C_n, for if it were then $x^r\sim_{\eta/3}0$; but by a δ_k-projection into C of the $\eta/3$-complex bounded by x^r we could obtain an η-complex in C bounded by $x_{j(k)}^r$, contrary to our hypothesis. This shows that ξ_n^r is essential for sufficiently large n and concludes the proof.

The necessity of the regular convergence in the preceding theorem is shown by the following example. Let A_i be the arc of the circle $\rho=1$ where θ varies from $1/i$ to $2\pi-(1/i)$; then A will be the circle $\rho=1$, and the 0-regular convergence is clearly violated. Now A is the smallest carrier of an essential 1-cycle, but the theorem cannot be satisfied, as A_i contains no essential 1-cycles.

COROLLARY 3.1. *If $A_i\to\leqq(r-1)A$, where A_i is a T_r-set* [5] *for each i, then A is a T_r-set.*

PROOF. If the theorem were not true, then T would contain an essential r-dimensional cycle; but by the theorem there would exist essential r-dimensional cycles in some of the A_i, which contradicts their property of being T_r-sets.

THEOREM 4. *If $M_i\to\leqq rM$ and B is an irreducible membrane* [2] *for the homology $\xi^r=(x_j^r)\sim 0$ in M, then B can be expressed as $\lim_{i\to\infty}B_i=B$, where $B_i\subset M_i$ is an irreducible membrane for a homology $\xi_i^r\sim 0$ in M_i.*

PROOF. Let $\epsilon_k\to 0$ be a sequence of positive numbers and let $3\delta_k=\delta^r(\epsilon_k)$ and $N^r(\epsilon_k)$ be the numbers corresponding to ϵ_k in Theorem

2. Choose N_k' such that for $i > N_k'$, $U_{\delta_k}(M_i) \supset M$, $U_{\delta_k}(M) \supset M_i$, and let $N_k'' = N^r(\epsilon_k) + N_k'$. This defines a number N_k'' for each k, and by letting $N_k = \sum_{j=1}^{k} N_j''$ we obtain a monotone increasing sequence of numbers. Since B is an irreducible membrane of $\xi^r \sim 0$ in M, we can find a V-chain $\eta^{r+1} = (y_j^{r+1})$ in M such that $\dot{y}_j^{r+1} = x_j^r$ for each j, and such that the point sets y_j^{r+1} converge to B. Let $j(k)$ be chosen such that $y_{j(k)}^{r+1}$ is a δ_k-complex and $U_{\epsilon_k}(y_{j(k)}^{r+1}) \supset B$, $U_{\epsilon_k}(B) \supset y_{j(k)}^{r+1}$. Let n be any fixed number greater than N_k, and project $y_{j(k)}^{r+1}$ by means of a δ_k-projection into a $(3\delta_k)$-complex y^{r+1} of M_n. Since $3\delta_k = \delta^r(\epsilon_k)$, each simplex of y^{r+1} has a V-chain realization in a subset of M_n of diameter less than ϵ_k. Combining these realizations for all simplices of y^{r+1}, we obtain a V-chain realization $\eta_n^{r+1} = (y_{jn}^{r+1})$ of y^{r+1} in $U_{\epsilon_k}(y^{r+1})$. Now a V-cycle can be formed from a subsequence of $(\dot{y}_{jn}^{r+1})$, and a further subsequence can be chosen so that the remaining (y_{jn}^{r+1}) converge to a set B_n'. We shall denote this subsequence by the same notation $\eta_n^{r+1} = (y_{jn}^{r+1})$ and we shall let $\xi_n^r = (\dot{y}_{jn}^{r+1})$. Let $B_n \subset B_n'$ be an irreducible membrane of the homology $\xi_n^r \sim 0$ in M_n. Now $U_\epsilon(B) \supset y_{j(k)}^{r+1}$, $U_{\delta_k}(y_{j(k)}^{r+1})^r \supset y^{r+1}$, $\overline{U}_{\epsilon_k}(y^{r+1}) \supset B_n' \supset B_n$, $B_n \supset y^{r+1}$ $U_{\delta_k}(y^{r+1}) \supset y_{j(k)}^{r+1}$, $U_{\delta_k}(y_{j(k)}^{r+1}) \supset B$; therefore $U_{3\epsilon_k}(B) \supset B_n$ and $U_{3\epsilon_k}(B_n) \supset B$. Thus if we choose ξ_n^r and B_n in M_n corresponding to ϵ_k for all n such that $N_k \leqq n < N_{k+1}$, we shall have $\lim_{n\to\infty} B_n = B$, and the conclusion of the theorem.

The necessity of the regular convergence in the preceding theorem is shown by allowing M_i to be a totally disconnected set for each i such that $\lim_{i\to\infty} M_i = M$ is a unit interval, and hence the irreducible membrane of the homology of the 0-cycle consisting of its end points. Now clearly the convergence is not 0-regular, and the conclusion of the theorem is violated since no 0-cycle of any M_i is ~ 0.

COROLLARY 4.1. *If $M_i \to \leqq rM$ and $A_i \to A$ where A_i is an A_r-set* [5] *of M_i for each i, then A is an A_r-set of M.*

PROOF. Consider any irreducible membrane B of the homology $\xi^r \sim 0$ in M where $\xi^r \subset A$. By Theorem 4 there exist cycles ξ_i^r in M_i for each i and irreducible membranes (B_i) of the homologies $\xi_i^r \sim 0$ in M_i, such that $B_i \to B$. Also since $A_i \to A$ and $\xi^r \subset A$, we can choose a ξ_i^r in A_i for each i. By the definition of an A_r-set we have $B_i \subset A_i$ for each i; therefore $B = \lim_{i\to\infty} B_i \subset \lim_{i\to\infty} A_i = A$. Thus A is an A_r-set of M.

THEOREM 5. *If $C_i \to \leqq rC$, where C_i is the irreducible carrier of an r-dimensional projection cycle ξ_i^r for each i, then C is the irreducible carrier of a projection cycle ξ^r. Finally ξ^r will be essential if and only if all but a finite number of the ξ_i^r are essential.*

PROOF. Corresponding to $\delta_i \to 0$, we can pick a subsequence of (C_i), which we shall suppose to be the whole sequence, such that $U_{\delta_i/3}(C_i) \supset C$, $U_{\delta_i/3}(C) \supset C_i$. Let $\xi_i^r = (x_{ij}^r)$; then there exists a $\delta_i/3$-cycle $x_{in_i}^r$ such that $U_{\delta_i/3}(x_{in_i}^r) \supset C_i$, $U_{\delta_i/3}(C_i) \supset x_{in_i}^r$. Project $x_{in_i}^r$ into a δ_i-cycle x_i^r of C. Clearly $U_{\delta_i}(x_i^r) \supset C$, $U_{\delta_i}(C) \supset x_i^r$. Pick a subsequence of (x_i^r), which we shall suppose to be the whole sequence, forming a V-cycle ξ^r. Now the point sets $x_i^r \to C$ and $\xi^r = (x_i^r) \subset C$; therefore C is the irreducible carrier of ξ^r.

Now suppose that ξ^r is not essential, then $\xi^r \sim 0$ in C. Since $C_i \to \leqq rC$, we know by a theorem of G. T. Whyburn [7] that C is an lc^r; hence by a theorem of R. L. Wilder [3] the r-dimensional Betti number of C is a finite number n. A result of H. A. Arnold [8] implies that all but a finite number of the C_i have this same finite Betti number. Let $\xi_{10}^r, \cdots, \xi_{n0}^r$ be a basis for r-dimensional cycles in C, which we can choose to be projection cycles with smallest carriers $C_{01}, \cdots, C_{0n}$. By Theorem 3 there exist cycles $\xi_{1i}^r, \cdots, \xi_{ni}^r$ with smallest carriers $C_{1i}, \cdots, C_{ni}$ in C_i such that $\lim_{i \to \infty} C_{ji} = C_j$ $(j=1, \cdots, n)$. The (ξ_{ji}^r) will be linearly independent for i greater than some integer N_1, for if not we can establish by a projection a linear dependence of the (ξ_{j0}^r). Since the Betti number of each C_i for i greater than some number N_2 is n, it follows that (ξ_{ji}^r) is a basis for cycles in C_i for $i > N = N_1 + N_2$. Thus $\xi_i^r \sim \sum_{j=1}^n \alpha_j \xi_{ji}^r$ $(i > N)$ $(\alpha_j = 0$ or $1)$, where we can suppose that N was chosen large enough so that the same linear combination holds for each i. Now by projecting the complexes bounded by $\xi_i^r + \sum_{j=1}^n \alpha_j \xi_{ji}^r$, we can establish a homology $\xi^r \sim \sum_{j=1}^n \alpha_j \xi_j^r$. But $\xi^r \sim 0$; therefore $\alpha_j = 0$ for all j. Thus $\xi_i^r \sim 0$ in C_i for $i > N$, which implies that ξ_i^r is inessential.

Conversely, if ξ^r is essential then exactly the same procedure as was used in Theorem 3 can be used to show that all but a finite number of the ξ_i^r are essential.

The necessity of the regular convergence in the preceding theorem is shown by the following example. Let C_i be the collection of points $(x = j/3^i, y = 0)$ for $j = 0, 1, \cdots, 3^i$, then $\lim_{i \to \infty} C_i = C =$ the unit interval from 0 to 1, and clearly the 0-regular convergence is violated. Now each C_i is an essential 0-dimensional V-cycle and hence is its own irreducible carrier, but C clearly cannot be the irreducible carrier of an essential 0-cycle as all 0-cycles are ~ 0 in C.

THEOREM 6. *If $B_i \to \leqq rB$, where B_i is an irreducible membrane for an homology of a projection cycle $\xi_i^r \sim 0$ in B_i for each i, then B is an irreducible membrane of an homology of a projection cycle $\xi^r \sim 0$ in B.*

PROOF. In the proof of Theorem 5 we have seen how to establish a projection cycle ξ^r in B. Furthermore by projections into B of the chains bounded by the ξ_i^r, we can establish an homology $\xi^r \sim 0$ in B. It remains to show that ξ^r is not ~ 0 in a proper subset B' of B. To this end suppose $\xi^r \sim 0$ in B'. By Theorem 4, $B' = \lim_{i\to\infty} B_i'$ where B_i' is an irreducible membrane of the homology $\xi_i^r \sim 0$ in B_i. (Since the carrier of ξ^r was chosen as the limit of the carriers of the ξ_i^r, we can choose the B_i' corresponding to the homologies of our original ξ_i^r.) Thus $B' = \lim_{i\to\infty} B_i' = \lim_{i\to\infty} B_i = B$ and B is an irreducible membrane of $\xi^r \sim 0$ in B.

THEOREM 7. *If $M_i^r \to \leqq rM$, where M_i^r is an r-dimensional closed Cantorian manifold* [6] *for each i, then if* dim $M \leqq r$, *M is also a closed r-dimensional Cantorian manifold.*

PROOF. Since $p^r(M_i^r) \neq 0$ for each i ($p^r(M_i^r)$ = the rth dimensional Betti number of M_i^r) and the convergence is regular, it follows that $p^r(M) \neq 0$. Next suppose M' is a proper closed subset of M with $p^r(M') \neq 0$. Then there exists an essential (projection) cycle ξ^r with irreducible carrier C in M'. By Theorem 3, $C = \lim_{k\to\infty} C_k$, where C_k is the smallest carrier of an essential cycle ξ_k^r in M_k^r. Now for some integer n, C_n will be a proper closed subset of M_n^r, but by the definition of M_n^r, $p^r(C_n) = 0$ contrary to the fact that ξ_n^r is essential. Thus $p^r(M') = 0$ for every proper closed subset M' of M. Finally dim $M \geqq r$, for $p^r(M) \neq 0$, implies the existence of an essential ξ^r in M. Since by hypothesis dim $M \leqq r$, we have dim $M = r$, and M is an r-dimensional closed Cantorian manifold.

BIBLIOGRAPHY

1. G. T. Whyburn, *On sequences and limiting sets*, Fund. Math. vol. 25 (1935) pp. 408–426.
2. R. L. Wilder, *Unified analysis situs*, Bull. Amer. Math. Soc. vol. 38 (1932) pp. 649–692.
3. ———, *On locally connected spaces*, Duke Math J. vol. 1 (1935) pp. 543–555.
4. P. Alexandroff and H. Hopf, *Topologie*, Berlin, 1935, p. 199.
5. G. T. Whyburn, *Cyclic elements of higher order*, Amer. J. Math. vol. 56 (1934) pp. 133–146.
6. P. Alexandroff, *Gestalt und Lage*, Ann. of Math. vol. 30 (1928) pp. 101–187.
7. G. T. Whyburn, *Regular convergence and monotone transformations*, Amer. J. Math. vol. 57 (1935) pp. 902–906.
8. H. A. Arnold, *On r-regular convergence of sets*, Bull. Amer. Math. Soc. Abstract 47-7-326.

LOUISIANA STATE UNIVERSITY