ON THE EXTENSION OF DIFFERENTIABLE FUNCTIONS

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The author has shown previously how to extend the definition of a function of class C^m defined in a closed set A so it will be of class C^m throughout space (see [1]).[1] Here we shall prove a uniformity property: If the function and its derivatives are sufficiently small in A, then they may be made small throughout space. Besides being bounded, we assume that A has the following property:

(P) There is a number ω such that any two points x and y of A are joined by an arc in A of length less than or equal to ωr_{xy} (r_{xy} being the distance between x and y).

This property was made use of in [2]; its necessity in the theorem is shown by two examples below.

A second theorem removes the boundedness condition in the first theorem, and weakens the hypothesis (P); its proof makes use of the proof of the first theorem. We remark that in each theorem, as in [1], the extended function is a linear functional of its values in A.

The proof of Theorem I is obtained by examining the proof in [1]; hence we assume that the reader has this paper before him, and we shall follow its notations closely.

THEOREM I. *Let A be a bounded closed set in n-space E with the property* (P), *and let m be a positive integer. Then there is a number α with the following property. Let $f(x)$ be any function of class C^m in A, with derivatives $f_k(x)$ ($\sigma_k = k_1 + \cdots + k_n \leqq m$). Suppose*

$$|f_k(x)| < \eta \qquad (x \in A, \sigma_k \leqq m).$$

Then $f(x)$ may be extended throughout E so that

$$|f_k(x)| < \alpha\eta \qquad (x \in E, \sigma_k \leqq m).$$

Let d be the diameter of A, or 1 if this is larger, and let R be a spherical region of radius $2d$ with its center at a point of A. Set $f(x) = 0$ in $E - R$. Then the extension of f in $R - A$ given in [1] will be shown to have the property, using

$$\alpha = 2n(m!)^n(m + 1)^{3n}(433n^{1/2}d\omega)^m cN,$$

where N and c are as given in [1, §§11, 12]. Note that $433 = 4 \cdot 108 + 1$.

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[1] Numbers in brackets refer to the references cited at the end of this paper.

Set $B=A\cup(\bar{R}-R)$. We show first that for any points x', x'' of B,

$$|R_k(x';x'')|<\beta r_{x'x''}^{m-\sigma_k}\eta,\qquad \beta=2n(m+1)^n\omega^m.$$

Suppose first that x' and x'' are in A. Let C be a curve in A joining them, of length less than or equal to $\omega r_{x'x''}$. The inequality is then a consequence of [2, Lemma 3]. Suppose next that one of the points is in A, and the other is in $\bar{R}-R$ (the case that both are in $\bar{R}-R$ is trivial). By [1, (3.1)], since $r_{x'x''}\geqq d\geqq 1$,

$$|R_k(x';x'')|\leqq \eta+\sum_{\sigma_l\leqq m-\sigma_k}\eta r_{x'x''}^{\sigma_l}\leqq r_{x'x''}^{m-\sigma_k}\eta\left[1+\sum_{\sigma_l\leqq m-\sigma_k}1\right]$$

$$\leqq (m+1)^n r_{x'x''}^{m-\sigma_k}\eta.$$

Now take any x in $R-B$. Let $\delta^*/4$ be the distance from x to B, and let x^* be a point of B distant $\delta^*/4$ from x. Say x is in the cube C of the set of cubes K_s; let $I_{\lambda_1},\cdots,I_{\lambda_t}$ be those I_λ with points in C (see [1, §11]). Now y^ν is the center of I_ν, and x^ν is a nearest point of B to y^ν. As noted in [1, (9.1)], $r_{y^\nu x^*}$ and $r_{y^\nu x^\nu}$ each lie between $\delta^*/8$ and $\delta^*/2$. Since $r_{xy^\nu}<\delta^*/2$, we have

$$r_{x^\nu x^*}<\delta^*,\qquad r_{xx^\nu}<\delta^*.$$

The definition of ζ in [1, §11] together with [1, (6.3)] gives

$$\zeta_{\nu;k}(x)=\psi_k(x;x^\nu)-\psi_k(x;x^*)=\sum_{\sigma_l\leqq m-\sigma_k}\frac{R_{k+l}(x^\nu;x^*)}{l!}(x-x^\nu)^l.$$

Hence

$$|\zeta_{\nu;k}(x)|<(m+1)^n\beta r_{x^\nu x^*}^{m-\sigma_k-\sigma_l}r_{xx^\nu}^{\sigma_l}\eta<(m+1)^n\beta\delta^{*m-\sigma_k}\eta.$$

Following [1, §11] still, we find

$$|D_kg(x)-\psi_k(x;x^*)|<c\sum_{\sigma_l\leqq m-\sigma_k}(m!)^n2^{s\sigma_l}N(m+1)^n\beta\delta^{*m-\sigma_k+\sigma_l}\eta.$$

As in [1], $2^s<108n^{1/2}/\delta^*$; hence

$$|D_kg(x)-\psi_k(x;x^*)|<c(m!)^nN(m+1)^{2n}(108n^{1/2})^m\beta\delta^{*m-\sigma_k}\eta.$$

Moreover, since $r_{xx^*}<3d$, [1, (6.1)] gives

$$|\psi_k(x;x^*)|<3^m(m+1)^nd^m\eta.$$

Since $\delta^*\leqq 4d$ and $f(x)=g(x)$ in $R-B$, the theorem follows.

We turn now to the second theorem. We shall say A *satisfies* (P) *locally* if for each $x\in A$ there is a neighborhood U of x and a number

ω such that any two points y and z of $A \cap U$ are joined by an arc in A of length not greater than ωr_{xy}.

THEOREM II. *Let A be a closed subset of an open set R in E, satisfying (P) locally, and let m be a positive integer. Then for any continuous function $\epsilon(x)$ defined and greater than 0 in R there is a continuous function $\delta(x)$ defined and greater than 0 in A with the following property. Let $f(x)$ be any function of class C^m in A, such that*

$$|f_k(x)| < \delta(x) \qquad (x \in A, \sigma_k \leqq m).$$

Then $f(x)$ may be extended throughout R so that

$$|f_k(x)| < \epsilon(x) \qquad (x \in R, \sigma_k \leqq m).$$

REMARKS. The preceding theorem is easily seen to be a consequence of this one. The present theorem holds if E is replaced by a differentiable manifold M, in which a fixed set of coordinate systems (each one intersecting but a finite number of others) is used to measure the size of derivatives. To show this, we imbed M in a Euclidean space E' (see [3, Theorem 1]), giving $A \subset R \subset R' \subset E'$ (R' open in E'; we let R' contain no points of the limit set of M), extend f throughout R' (see the proof of [3, Lemma 4]), and consider its values in R.

To prove the theorem, we begin by choosing spherical regions $U_1, U_2, \cdots$, each $\overline{U}_i \subset R$, with the following properties:

(a) Each U_i is in a neighborhood U as described above.

(b) Each $\overline{U}_i$ intersects but a finite number of other $\overline{U}_j$.

(c) If U_i is of radius ρ_i, and U_i' is the concentric region of radius $\rho_i/2$, then $R = \sum U_i'$.

Let $\psi^i(x)$ be a function of class C^m in E such that

$$\psi^i(x) > 0 \qquad (x \in U_i'),$$
$$\psi^i(x) = 0 \qquad (x \in E - U_i').$$

Set

$$\phi^i(x) = \psi^i(x) / \sum \psi^j(x) \qquad (x \in R);$$

then $\phi^i(x)$ is of class C^m in R, and

$$\phi^i(x) = 0 \qquad (x \in R - U_i'),$$
$$\sum \phi^i(x) = 1 \qquad (x \in R).$$

The extension of $f(x)$ is defined as follows. Set

$$f^i(x) = \phi^i(x) f(x) \qquad (x \in A),$$
$$f^i(x) = 0 \qquad (x \in R - U_i).$$

Then f^i is of class C^m in $A \cup (R - U_i)$. Extend it as in [1] (using a fixed subdivision of $U_i - A$; we could set $f^i(x)=0$ in $E-R$) to be of class C^m in R (or E). (Note that if $A \cap U_i' = 0$, then $f^i(x)=0$, $x \in R$.) Set

$$f(x) = \sum f^i(x) \qquad (x \in R).$$

Then f is an extension of class C^m of its values in A. We must show that it satisfies the condition of smallness.

Choose $a_i \geqq 1$ so that

$$\left| \overset{i}{\phi}_k(x) \right| \leqq a_i \qquad (x \in R,\ \sigma_k \leqq m),$$

then if $|f_k(x)| < \eta$ $(x \in A \cap U_i')$,

$$\left| \overset{i}{f}_k(x) \right| = \left| \sum_{\sigma_l \leqq \sigma_k} \overset{i}{\phi}_l(x) f_{k-l}(x) \right| \leqq (m+1)^n a_i \eta \qquad (x \in A).$$

By the choice of U_i, there is an ω_i such that any x' and x'' in $A \cap U_i$ are joined by an arc in A of length not greater than $\omega_i r_{x'x''}$. Set $\sigma_i = \max(1, 2/\rho_i)$. If $\overset{i}{R}_k$ is the remainder for $\overset{i}{f}_k$, we shall show that for any x' and x'' in $A \cup (R - U_i)$,

$$\left| \overset{i}{R}_k(x'; x'') \right| < 2n(m+1)^{2n} \omega_i^m a_i \sigma_i^m r_{x'x''}^{m-\sigma_k} \eta.$$

If x' and x'' are both in U_i, we apply [2, Lemma 3]. If $x' \in R - U_i$ and $x'' \in U_i'$, or vice versa, then $r_{x'x''} \geqq \rho_i/2$, and the proof in the preceding theorem applies; we consider separately the cases $\rho_i/2 \geqq 1$, $\rho_i/2 < 1$, using $r_{x'x''} \geqq 1$ and $\sigma_i r_{x'x''} \geqq 1$ respectively. If $x' \in R - U_i$ and $x'' \in R - U_i'$, or vice versa, $\overset{i}{R}_k = 0$, since $\overset{i}{\phi}_l(x') = \overset{i}{\phi}_l(x'') = 0$. The proof of the preceding theorem now shows that for some α_i, if

$$\left| f_k(x') \right| < \eta \qquad (x' \in A \cap U_i',\ \sigma_k \leqq m),$$

then

$$\left| \overset{i}{f}_k(x) \right| < \alpha_i \eta \qquad (x \in R,\ \sigma_k \leqq m).$$

(We may set $\alpha_i = 1$ if $A \cap U_i' = 0$.)

Given $\epsilon(x)$, we determine $\delta(x)$ as follows. For each $x \in R$ there is a set of numbers $\lambda_1, \cdots, \lambda_s$, $s = s(x)$, such that

$$x \in \text{each } U'_{\lambda_j}, \qquad x \in \text{no other } U_l'.$$

Because of (b), s is finite. Set $\alpha(x) = \alpha_{\lambda_1} + \cdots + \alpha_{\lambda_s}$.
We can clearly choose a continuous function $\beta(x)$ in R such that

$$\alpha(x) < \beta(x) \qquad (x \in R).$$

We may now choose a continuous function $\delta(x') > 0$ in A such that for any $x'\in A$, if $U'_{\mu_1},\cdots,U'_{\mu_t}$ are the U'_j containing x', then

$$\delta(x') \leqq \epsilon(x)/\beta(x) \quad (x\in U'_{\mu_1}\cup\cdots\cup U'_{\mu_t}).$$

Now take any f of class C^m in A, with $|f_k(x)|<\delta(x)$ $(x\in A, \sigma_k\leqq m)$; the extension of f through R has been defined. Take any $x\in R$; define $\lambda_1,\cdots,\lambda_s$ as above. Then

$$|f_k(x')| < \delta(x') \leqq \epsilon(x)/\beta(x) \qquad (x'\in A\cap U'_{\lambda_j}, \sigma_k\leqq m),$$

and hence

$$|f_k^{\lambda_j}(x)| < \alpha_{\lambda_j}\epsilon(x)/\beta(x) \qquad (\sigma_k\leqq m).$$

Since $f_k(x)=f_k^{\lambda_1}(x)+\cdots+f_k^{\lambda_s}(x)$,

$$|f_k(x)| < \alpha(x)\epsilon(x)/\beta(x) < \epsilon(x)$$

for $\sigma_k\leqq m$, which completes the proof.

EXAMPLES. (1) Let A consist of a point, together with a sequence of points approaching it. Letting $f(x)=1$ at a finite number of points of the sequence, and $f(x)=0$ in the rest of A shows (with $m=1$) that the theorem fails here.

(2) Let A be the closed region of the plane defined by (a) $x^2+y^2\leqq 1$, and (b) either $x\leqq 0$ or $|y|\geqq x^{3/2}$. Let $f(x, y)=0$ if $x\leqq 0$, and set

$$f(x,y)=\begin{cases}\gamma x^2/(1+\gamma^2x^2) & \text{if} \quad x\geqq 0,\ y>0,\\ -\gamma x^2/(1+\gamma^2x^2) & \text{if} \quad x\geqq 0,\ y<0.\end{cases}$$

We see easily that f is of class C^1 in A. (It would not be if, in (b), we used $|y|\geqq x^2$.) The maximum $\partial f/\partial x$ occurs at $x=1/(3^{1/2}\gamma)$, and has the value $9/(8\cdot 3^{1/2})$. Set

$$p=(1/3^{1/2}\gamma,\ 1/3^{3/4}\gamma^{3/2}), \qquad q=(1/3^{1/2}\gamma,\ -1/3^{3/4}\gamma^{3/2}).$$

Then

$$\frac{f(p)-f(q)}{r_{pq}}=\frac{2\gamma/3\gamma^2}{1+\gamma^2/3\gamma^2}\div\frac{2}{3^{3/4}\gamma^{3/2}}=\frac{3^{3/4}}{4}\gamma^{1/2}.$$

Hence, in any extension of f through the plane, we must have $|\partial f/\partial y|\geqq 3^{3/4}\gamma^{1/2}/4$ at some point (between p and q); yet $|f|$, $|\partial f/\partial x|$ and $|\partial f/\partial y|$ are uniformly bounded for all $\gamma>1$. Taking γ arbitrarily large shows that the conclusion of the theorem does not hold.

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THE SYMMETRIC JOIN OF A COMPLEX

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1. **The definition of J.** Let K be a polyhedron. With each pair of distinct points p, q of K we associate a closed line segment pq. No distinction is made between p and q and the corresponding end points of pq. The length of pq is a continuous function of p and q, and the length approaches zero if p and q approach a common limit. Distinct segments do not intersect except at a common end point. The points of these segments with their obvious natural topology make up J, the symmetric join of K. This space arises in [4][1] in connection with the problem of finding the chords of a manifold that are orthogonal to the manifold.

2. **The subdivision of J.** Let the mid-point of pq be denoted by $\Lambda p\times q=\Lambda q\times p$, and let $p=\Lambda p\times p$. These points $\Lambda p\times q$ make up the symmetric product S of K. Let the mid-point of the segment from p to $\Lambda p\times q$ be denoted by $p\times q$, and let $p=p\times p$. These points $p\times q$ make up the topological product $P=K\times K$. Consider the closed segment of pq from $p\times q$ to $q\times p$, it being understood that this segment is the point p when $p=q$. All such segments form the "neighborhood" N_S. Clearly N_S can be homotopically deformed in N_S along the segments pq upon S with S remaining pointwise invariant. Finally consider the closed segment of pq from p to $p\times q$, it being understood that this segment is the point p when $p=q$. All such segments form the "neighborhood" N_K. Clearly N_K can be homotopically deformed in N_K along the segments pq upon K with K remaining pointwise invariant.

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[1] Numbers in brackets refer to the References at the end of the paper.