ON THE SURFACES OF COINCIDENCE

SU-CHENG CHANG

The purpose of this note is to give a characteristic property of the surfaces of coincidence. We show first that there are six and only six asymptotic osculating quadrics Q_j ($j=1, \cdots, 6$) associated with a point P of a non-ruled analytic surface (M) each of which contains the four consecutive asymptotic tangents $t_{i(j)}$ ($i=1, 2, 3, 4$) of one system along a curve T through P and on the surface (M). As the curve T must be tangent to a direction of Darboux at P, it is further proved that each quadric Q_j is an osculating quadric of an asymptotic ruled surface R along a curve of Darboux when and only when the surface (M) is a surface of coincidence.

Let the surface (M) be referred to its asymptotic parameters u, v, and let $\{M, M_1, M_2, M_3\}$ be the normal tetrahedron of Cartan, M_1M and M_2M being the two asymptotic tangents and MM_3, M_1M_2 the directrices of Wilczynski. Then the surface (M) is, except for a projective transformation, determined by the following system of equations:

$$
(1)\quad \begin{cases}
\dfrac{\partial M}{\partial u} = \dfrac{1}{2}\,\dfrac{\partial \log \gamma}{\partial u} M + M_1, \quad \dfrac{\partial M_1}{\partial u} = B^2 M - \dfrac{1}{2}\,\dfrac{\partial \log \gamma}{\partial u} M_1 + \beta M_2, \\[2ex]
\dfrac{\partial M_2}{\partial u} = \dfrac{h_1}{2} M + \dfrac{1}{2}\,\dfrac{\partial \log \gamma}{\partial u} M_2 + M_3, \\[2ex]
\dfrac{\partial M_3}{\partial u} = A^2 \beta M + \dfrac{1}{2} h_1 M_1 + B^2 M_2 + \dfrac{-1}{2}\,\dfrac{\partial \log \gamma}{\partial u} M_3; \\[2ex]
\dfrac{\partial M}{\partial v} = \dfrac{1}{2}\,\dfrac{\partial \log \beta}{\partial v} M + M_2, \quad \dfrac{\partial M_2}{\partial v} = A^2 M - \dfrac{1}{2}\,\dfrac{\partial \log \beta}{\partial v} M_2 + \gamma M_1, \\[2ex]
\dfrac{\partial M_1}{\partial v} = \dfrac{k_1}{2} M + \dfrac{1}{2}\,\dfrac{\partial \log \beta}{\partial v} M_2 + M_3, \\[2ex]
\dfrac{\partial M_3}{\partial v} = B^2 \gamma M + A^2 M_1 + \dfrac{1}{2} k_1 M_2 - \dfrac{1}{2}\,\dfrac{\partial \log \beta}{\partial v} M_3;
\end{cases}
$$

where M denotes a point of the surface with coordinates M^i ($i=1, 2, 3, 4$) and

$$h_1 = \beta\gamma - (\log \beta)'', \qquad k_1 = \beta\gamma - (\log \gamma)''.$$

The integrability conditions of the system (1) are

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$$(2)\qquad \frac{\partial A^2}{\partial u}=\frac{h_1}{2}\frac{\partial(\log\beta h_1)}{\partial v},\qquad \frac{\partial B^2}{\partial v}=\frac{k_1}{2}\frac{\partial(\log\gamma k_1)}{\partial u},\qquad A\frac{\partial(A\beta)}{\partial v}=B\frac{\partial(B\gamma)}{\partial u}.$$

Following Godeaux,[1] we denote by $\cdots, U_n, \cdots, U_1, U, V, V_1, \cdots, V_n, \cdots$ the self-polar Laplace sequence in S_5, where $U=|M, M_1|$ and $V=|M, M_2|$ are the images in S_5 of the two asymptotic tangents of the surface (M) at M respectively. It is well known that[2]

$$(3)\qquad U^{10}=\beta V,\qquad V^{01}=\gamma U;$$

$$(4)\qquad \begin{cases} U_n^{01}=U_{n+1}+U_n(\log\beta h_1\cdots h_n)^{01},\\ U_n^{10}=h_nU_{n-1},\\ h_n=h_{n-1}-(\log\beta h_1\cdots h_n)^{11};\end{cases}$$

and

$$(5)\qquad \begin{cases} V_n^{10}=V_{n+1}+V_n(\log\gamma k_1\cdots k_n)^{10},\\ V_n^{01}=k_nV_{n-1},\\ k_n=k_{n-1}-(\log\gamma k_1\cdots k_n)^{11},\end{cases}$$

where $(\quad)^{ij}$ denotes the partial derivative of order $i+j$ formed by differentiating $(\quad)$ i times with respect to u and j times with respect to v.

Since seven points in S_5 are always dependent, we find that

$$(6)\qquad \begin{aligned} &U_3+(\log\beta^3h_1^2h_2)^{01}U_2+\big[(\log\beta h_1)^{02}-4A^2\\ &\quad+(\log\beta h_1)^{01}(\log\beta^2h_1)^{01}\big]U_1\\ &\quad-4A^2(\log A\beta)^{01}U+4\gamma B^2V-\gamma(\log\gamma k_1)^{10}V_1-\gamma V_2=0\end{aligned}$$

and

$$(7)\qquad \begin{aligned} &V_3+(\log\gamma^3k_1^2k_2)^{10}V_2+\big[(\log\gamma k_1)^{20}-4B^2\\ &\quad+(\log\gamma k_1)^{10}(\log\gamma^2k_1)^{10}\big]V_1\\ &\quad-4B^2(\log B\gamma)^{10}V+4\beta A^2U-\beta(\log\beta h_1)^{01}U_1-\beta U_2=0.\end{aligned}$$

[1] See Godeaux, *La theorie des surfaces et l'espace réglé*, 1934, Paris. A direct definition of the sequence of Godeaux quadrics has been given by the present author. See S. C. Chang, *Some theorems on ruled surfaces*, in the press of Science Records, Academia Sinica.

[2] L. Godeaux, loc. cit.

If four consecutive asymptotic u-tangents ($v=$const.) belong to a regulus, then there is a plane in S_5 which intersects the surface (U) at four consecutive points. In other words: U, dU, d^2U, d^3U should be coplanar.

By means of (3), (4), (5) and (6) it is clear that the three points

$$\beta V du + U_1 dv, \tag{8}$$

$$\begin{aligned} V[(\beta^{10} + \beta(\log \gamma)^{10})(du)^2 + \beta^{01} du dv + \beta d^2 u] \\ + \beta V_1 (du)^2 + U_1 d^2 v + [U_2 + U_1 (\log \beta h_1)^{01}](dv)^2 \end{aligned} \tag{9}$$

and

$$\begin{aligned} &[U_3 - 4A^2(\log A\beta)^{01} U](dv)^3 + MU_1 + \beta V_2 (du)^3 + NV \\ &+ V_1[\beta(\log \gamma^2 k_1)^{10}(du)^3 + 2\beta^{10}(du)^3 + 2\beta^{01} dv (du)^2 + 3\beta du d^2 u] \\ &+ U_2[(\log \beta^2 h_1^2 h_2)^{01}(dv)^3 + 3dv d^2 v] \end{aligned} \tag{10}$$

are collinear, where

$$\begin{aligned} M = {} & [(\log \beta h_1)^{01}]^2 (dv)^3 + d^3 v + h_1 du (dv)^2 + (\log \beta h_1)^{02}(dv)^3 \\ & + 3(\log \beta h_1)^{01} d^2 v dv, \\ N = {} & 3\beta^{10} du d^2 u + (du)^3[\beta^{20} + \beta((\log \gamma)^{10})^2 + \beta(\log \gamma)^{20} + 2\beta^{10}(\log \gamma)^{10}] \\ & + (du)^2 dv[2\beta^{11} + \beta(\log \gamma)^{11} + 2\beta^{01}(\log \gamma)^{10} + \beta k_1] + \beta^{02} du (dv)^2 \\ & + 2\beta^{01} d^2 u dv + \beta^{01} du d^2 v + \beta d^3 u + 3\beta(\log \gamma)^{10} d^2 u du. \end{aligned}$$

From (6), (8), (9) and (10) it follows that the coefficient of V_2 in (10) must vanish, so that

$$\beta(du)^3 + \gamma(dv)^3 = 0. \tag{11}$$

Moreover, we demand that the coefficients of U_2, V_1 in (9) should be proportional to those in (10), and therefore that

$$du d^2 v - dv d^2 u - (1/3) du dv[(\log \beta^2 \gamma)^{10} du + (\log \beta^3)^{01} dv] = 0. \tag{12}$$

Differentiation of (11) shows that

$$3(du d^2 v - dv d^2 u) + (du)^2 dv (\log (\gamma/\beta))^{10} + du (dv)^2 (\log (\gamma/\beta))^{01} = 0. \tag{13}$$

If (12) and (13) coincide with each other for any direction of Darboux, then

$$(\log \beta\gamma^2)^{10} = 0, \qquad (\log \gamma\beta^2)^{01} = 0. \tag{14}$$

Thus we may take

$$\beta = 1, \qquad \gamma = 1, \tag{15}$$

and accordingly,

$$N = d^3u + (du)^2dv, \qquad M = d^3v + du(dv)^2.$$

Hence the surface in consideration must be a surface of coincidence.

We inquire now whether the asymptotic tangents of some one system along a curve on a surface belong to a regulus. It is seen from (11) that the curve is necessarily a Darboux curve. In virtue of (11) and (15) we obtain a point

$$V[d^3u + (du)^2dv + (-4B^2)(dv)^3 - 3d^2vd^2u/dv] \\ + U_1[d^3v + du(dv)^2 + 4A^2(du)^3 - 3(d^2v)^2/dv].$$

This is collinear with the points (9) and (10) and coincides with the point (8) when

$$(16) \qquad dv[d^3u + (du)^2dv - 4B^2(dv)^3 - 3d^2vd^2u/dv] \\ - du[d^3v + du(dv)^2 + 4A^2(dv)^3 - 3(d^2v)^2/dv] = 0.$$

From (13), (15) and (16) we have

$$(17) \qquad A^2 = 0, \qquad B^2 = 0.$$

A reference to (15) and (17) shows that *the surface must be* $xyz=1$ *or one of its projective transforms.*

NATIONAL UNIVERSITY OF CHEKIANG