# UNIFORM CONVEXITY. III

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It is the purpose of this note to fill out certain results given in two recent papers on uniform convexity of normed vector spaces.[1] A normed vector space[2] $B$ is called *uniformly convex* with *modulus of convexity* $\delta$ if for each $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that for every two points $b$ and $b'$ of $B$ satisfying the conditions $\|b\| = \|b'\| = 1$ and $\|b - b'\| \geqq \epsilon$ the quantity $\|b + b'\| \leqq 2(1 - \delta(\epsilon))$. If $\|b_0\| = 1$, $B$ is said to be *locally uniformly convex near* $b_0$ if there is a sphere about $b_0$ in which the condition for uniform convexity holds. Theorem 1 shows that all properties of normed vector spaces which are invariant under isomorphism are the same for uniformly convex and locally uniformly convex spaces. Theorem 2 gives a necessary condition for isomorphism with a uniformly convex space. The condition is in terms of isomorphisms of finite dimensional subspaces and is suggested by examples given in [I]; it is not known whether the condition is sufficient. Theorem 3 is somewhat more general than Theorem 3 of [II]; it uses uniformly convex function spaces instead of the $l_p$ spaces of [II].

A *cone* $C$ in $B$ is a set which contains all of every half line from the origin through each point of $C$.

LEMMA 1. *A normed vector space $B$ is locally uniformly convex near $b_0$ if and only if there exists a convex cone $C$, with $b_0$ in its interior, such that for every $\epsilon$ there is a $\delta_1(\epsilon) > 0$ such that the conditions $\|b\| \leqq 1$, $\|b'\| \leqq 1$, and $\|b - b'\| \geqq \epsilon$ imply $\|b + b'\| \leqq 2(1 - \delta_1(\epsilon))$ for every pair of points $b$ and $b'$ in $C$.*

If this condition is satisfied there is obviously a sphere about $b_0$ inside $C$, so that in that sphere $\delta(\epsilon)$ can be taken equal to $\delta_1(\epsilon)$. On the other hand, if there is a sphere of radius $2k$ about $b_0$ in which $\delta$ can be defined, it can be shown that it suffices to let $C$ be the cone through points of the sphere of radius $k$ about $b_0$ and to let $\delta_1(\epsilon) = \inf [\epsilon/10, \delta(4\epsilon/5)/2]$.

LEMMA 2. *If the cone $C$ of Lemma 1 contains a sphere about $b_0$ of radius $k$, if $\|b\| \leqq 1$ and if $\|b - b_0\| \geqq k$, then $\|b + b_0\| < 2 - \delta_1(k)$.*

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[1] These papers are [I] *Reflexive Banach spaces not isomorphic to uniformly convex spaces*, Bull. Amer. Math. Soc. vol. 47 (1941) pp. 313–317, and [II] *Some more uniformly convex spaces*, Bull. Amer. Math. Soc. vol. 47 (1941) pp. 504–507.

[2] See Banach, *Théorie des opérations linéaires*, Warsaw, 1932, for general definitions.

This is obvious if $\|b-b_0\|=k$. If $\|b\|\leqq 1$ and $\|b-b_0\|>k$ there exists a point $b_1=\lambda b+(1-\lambda)b_0$, $0<\lambda<1$, on the line segment from $b_0$ to $b$ such that $\|b_0-b_1\|=k$ while $\|b_1\|\leqq 1$; hence $\|b_1+b_0\|\leqq 2(1-\delta_1(k))$. Let $f$ be a linear functional such that $f(b')\leqq\|b'\|$ for all $b'$ in $B$ and such that the line $\{b'|f(b')=1\}$ in the plane of 0, $b_0$ and $b$ touches the unit sphere in $B$ at the point of intersection of that sphere with the half line from 0 through $b+b_0$, so that $f(b+b_0)=\|b+b_0\|$. Then $f(b_0+b_1)\leqq\|b_0+b_1\|\leqq 2(1-\delta_1(k))$. Two cases can now be distinguished: If $f(b_0)\geqq f(b)$, $2-2\delta_1(k)\geqq f(b_1+b_0)\geqq f(b+b_0)=\|b+b_0\|$. If $f(b_0)<f(b)$, $2(1-\delta_1(k))\geqq f(b_1+b_0)=f(b_1)+f(b_0)>2f(b_0)$, so $\|b+b_0\|=f(b+b_0)=f(b_0)+f(b)<1-\delta_1(k)+1$.

THEOREM 1. *If $B$ is locally uniformly convex near some point $b_0$, then $B$ is isomorphic to a uniformly convex space. If $k$ is the radius of the sphere which exists by Lemma 1 about $b_0$, a suitable modulus of convexity for the new space is given in terms of the old by $\delta_1'(\epsilon)=1-1/[1+\delta_1(\delta_1(k)\epsilon/4)/(k+\delta_1(k)/4)]$.*

Suppose the cone $C$ of Lemma 1 contains a sphere $\{b|\,\|b-b_0\|\leqq k\}$ about $b_0$; let $\alpha=1-\delta_1(k)/4$ and consider the two spheres $E_1=\{b|\,\|b-\alpha b_0\|\leqq 1\}$ and $E_2=\{b|\,\|b+\alpha b_0\|\leqq 1\}$. If $S$ is the intersection of $E_1$ and $E_2$, it is clear that $S$ is convex and symmetric about the origin, and that $S$ contains the sphere $\{b|\,\|b\|\leqq\delta_1(k)/4\}$.

To show $\|b\|\leqq k+\delta_1(k)/4$ for each $b$ in $S$, it suffices to show that $b\in S$ implies that $b+\alpha b_0$ is within $k$ of $b_0$. If this is false, that is, if $\|b+\alpha b_0-b_0\|>k$, then, by Lemma 2, $\|b+\alpha b_0+b_0\|<2-\delta_1(k)$. However $\|b+\alpha b_0+b_0\|=\|b-\alpha b_0+(1+2\alpha)b_0\|\geqq(1+2\alpha)\|b_0\|-\|b-\alpha b_0\|\geqq 1+2\alpha-1=2\alpha=2-\delta_1(k)/2>2-\delta_1(k)$.

Let $|b|$ be the smallest non-negative value of $t$ for which the point $b/t$ is in $S$. Then $|\cdots|$ defines a new norm in $B$ and it is clear from the inequalities thus far derived that $[\delta_1(k)/4]|b|\leqq\|b\|\leqq[k+\delta_1(k)/4]|b|$, so this new norm defines a space isomorphic to the original and all that need be proved is that $|\cdots|$ is uniformly convex. If $b_1$, $b_2\in S$, and $|b_1-b_2|>\epsilon$, then $\|(b_1+\alpha b_0)-(b_2+\alpha b_0)\|=\|b_1-b_2\|\geqq\delta_1(k)\epsilon/4$. Also $\|b_i+\alpha b_0-b_0\|\leqq k$ by the preceding paragraph so, by the original hypotheses near $b_0$, $\|b_1+b_2+2\alpha b_0\|\leqq 2[1-\delta_1(\delta_1(k)\epsilon/4)]\equiv 2(1-\mu(\epsilon))$; that is, $(b_1+b_2)/2=E_1'=\{b\,|\,\|b+\alpha b_0\|\leqq 1-\mu(\epsilon)\}$. The same argument with $-\alpha b_0$ and $-b_0$ shows that $(b_1+b_2)/2\in E_2'=\{b|\,\|b-\alpha b_0\|\leqq 1-\mu(\epsilon)\}$.

It will now suffice to show that there is a $\delta_1'(\epsilon)>0$ such that $|b|<1-\delta_1'(\epsilon)$ if $b\in E_1'\cdot E_2'$. $E_i'\subset E_i$, $i=1, 2$, so for any $b$ in $E_1'\cdot E_2'$ there is a number $t\geqq 1$ such that $|tb|=1$; hence, either $\|tb+\alpha b_0\|=1$ or $\|tb-\alpha b_0\|=1$. These cases are interchanged by replacing $b$ by

$-b$ so it suffices to consider the first; then $1-\mu(\epsilon) \geqq \|\alpha b_0 - b\| = \|\alpha b_0 - tb + tb - b\| \geqq \|\alpha b_0 - tb\| - \|tb - b\| = 1-(t-1)\|b\|$. Therefore $(t-1)\|b\| \geqq \mu(\epsilon)$ or $t \geqq 1+\mu(\epsilon)/\|b\| \geqq 1+\delta_1(\delta_1(k)\epsilon/4)/[k+\delta_1(k)/4]$. Letting $1-\delta_1'(\epsilon)$ be the reciprocal of the last term in the preceding inequality gives $|b| = 1/t \leqq 1-\delta_1'(\epsilon)$ if $b \in E_1' \cdot E_2'$.

We turn now to a necessary condition for isomorphism of $B$ with a uniformly convex space. The effect of uniform convexity on the finite dimensional subspaces of an isomorphic space was used implicitly in [I]; it is given explicit formulation here. Let $B_0$ and $B$ be two normed vector spaces; then there exist linear operations of norm $\leqq 1$ defined on $B_0$ with values in $B$. For each such operator $U$ there is a largest number $k_U$, $0 \leqq k_U \leqq 1$, such that $\|b_0\| \geqq \|U(b_0)\| \geqq k_U\|b_0\|$ for each $b_0$ in $B_0$, and this number $k_U$ can be taken as a measure of the distortion of $B_0$ under the mapping $U$ into $B$. Define $k(B_0, B)$ to be the least upper bound of $k_U$ as $U$ runs over the linear operators from $B_0$ to $B$ of norm not greater than 1; explicitly, $k(B_0, B) = \sup_{\|U\| \leqq 1} \inf_{\|b_0\|=1} \|U(b_0)\|$. $k(B_0, B)$ is then a measure of how nearly $B_0$ approaches isometry with a subspace of $B$; if $k(B_0, B) = 1$, there are operations which come arbitrarily near preserving distances; $k(B_0, B) > 0$ if and only if $B_0$ is isomorphic to a subspace of $B$. For the present it suffices to choose certain finite dimensional spaces for $B_0$. Let $M_n$ and $L_n$ be the $n$-dimensional spaces of sequences $t = (t_1, \cdots, t_n)$ of $n$ real numbers, where $\|t\|_{M_n} = \|(t_1, \cdots, t_n)\|_{M_n} = \max_{1 \leqq i \leqq n} |t_i|$ and $\|t\|_{L_n} = \|(t_1, \cdots, t_n)\|_{L_n} = \sum_{1 \leqq i \leqq n} |t_i|$. Then $k(M_n, B) = k(L_n, B) = 0$ if and only if the dimension of $B$ is less than $n$; also $k(M_n, B)$ and $k(L_n, B)$ are nonincreasing functions of $n$ for each $B$.

LEMMA 3. *If $U$ is a one-to-one linear operator from $B_1$ onto $B_2$ such that for some $a \geqq 0$, $\|b_1\| \geqq \|Ub_1\| \geqq a\|b_1\|$ for each $b_1$ in $B_1$, then for any normed vector space $T$, $k(T, B_1) \geqq ak(T, B_2) \geqq a^2k(T, B_1)$.*

If $a = 0$, this is obvious. If $a > 0$ and $F$ is any linear operator from $T$ into $B$, with $\|F\| \leqq 1$, let $UF$ be defined by $UF(t) = U(F(t))$ for every $t$ in $T$. Then $\|UF\| \leqq 1$ and $\|UF(t)\| \geqq a\|F(t)\|$ for every $t$. Hence $\inf_{\|t\|=1} \|UF(t)\| \geqq a \inf_{\|t\|=1} \|F(t)\|$ so $k(T, B_2) \geqq ak(T, B_1)$. If $F'$ is any linear operator of norm $\leqq 1$ from $T$ into $B_2$, the same argument, using the operator $aU^{-1}F'$, shows that $k(T, B_1) \geqq ak(T, B_2)$.

Note that if $U$ maps $B_1$ on only part of $B_2$ or is not 1-1 but is of norm $\leqq 1$, the first half of the proof still holds (although $a = 0$ in the second case); it follows that if $B_1$ is a subspace of $B_2$, then $k(T, B_1) \leqq k(T, B_2)$.

Sobczyk[3] has defined a special embedding of $l_1$ into $m$ which can easily be modified to define an isometry of $L_{n+1}$ and a subspace of $M_{2^n}$ so $k(L_{n+1}, B) \geqq k(M_{2^n}, B)$ for every integer $n$. In particular, $L_2$ and $M_2$ are isometric so $k(L_2, B) = k(M_2, B)$.

LEMMA 4. *If $\delta_1$ satisfies Lemma 1 in the whole unit sphere of $B$ and is continuous on the left, then*
(1) $k(M_n, B) \leqq [1 - \delta_1(2k(M_n, B))]^{n-1}$,
(2) $k(L_{2^n}, B) \leqq [1 - \delta_1(2k(L_{2^n}, B))]^n$.

If $F$ is an operation from $M_n$ into $B$ such that $\|t\| \geqq \|F(t)\| \geqq k\|t\|$ for all $t$, where $k > 0$, let $\epsilon_i = \pm 1$ for $i = 1, \cdots, n$; then the points $F(\epsilon_1, \cdots, \epsilon_n)$ lie in the unit sphere of $B$ since $\|F(\epsilon_1, \cdots, \epsilon_n)\| \leqq \|\epsilon_1, \cdots, \epsilon_n\| = 1$. If $\epsilon_1, \cdots, \epsilon_n$ and $\epsilon_1', \cdots, \epsilon_n'$ are different, $\|F(\epsilon_1, \cdots, \epsilon_n) - F(\epsilon_1', \cdots, \epsilon_n')\| \geqq k\|(\epsilon_1, \cdots, \epsilon_n) - (\epsilon_1', \cdots, \epsilon_n)\| = 2k$; hence $\|F(\epsilon_1, \cdots, \epsilon_{n-1}, 0)\| = \|F(\epsilon_1, \cdots, \epsilon_{n-1}, 1) - F(\epsilon_1, \cdots, \epsilon_{n-1}, -1)\|/2 \leqq 1 - \delta_1(2k)$; that is, $\|F(\epsilon_1, \cdots, \epsilon_{n-1}, 0)/[1 - \delta_1(2k)]\| \leqq 1$ for all $\epsilon_1, \cdots, \epsilon_{n-1}$. These points are at least $2k$ apart for different $\epsilon_i$, so this process can be applied $n-1$ times to show that $\|F(1, 0, 0, \cdots, 0)/[1 - \delta_1(2k)]^{n-1}\| \leqq 1$. Hence $k = k\|1, 0, 0, \cdots, 0\| \leqq \|F(1, 0, 0, \cdots, 0)\| \leqq [1 - \delta_1(2k)]^{n-1}$. Taking $k = k(M_n, B)$ or, if that is impossible, taking the limit as $k$ increases toward $k(M_n, B)$ gives (1).

If $F$ maps $L_{2^n}$ into $B$ so that $\|t\| \geqq \|F(t)\| \geqq k\|t\|$, $k > 0$, for all $t$, the same sort of argument can be carried through using the points of $L_{2^n}$ which have one coordinate equal to one, the others all zero. It leads to the inequality $k = k\|(2^{-n}, \cdots, 2^{-n})\| \leqq \|F(2^{-n}, \cdots, 2^{-n})\| \leqq [1 - \delta_1(2k)]^n$ which gives (2).

THEOREM 2. *If $B$ is isomorphic to a space which is locally uniformly convex near any point, then* $\lim_n k(M_n, B) = \lim_n k(L_n, B) = 0$.

By Theorem 1, $B$ is isomorphic to a uniformly convex space $B'$. By Lemma 4, $k(L_{2^n}, B') < [1 - \delta_1(2k(L_{2^n}, B'))]^n$ for all $n$. If $k(L_{2^n}, B') > k > 0$ for all $n$, then $0 < k \leqq k(L_{2^n}, B') \leqq (1 - \delta_1(2k))^n \rightarrow 0$ as $n \rightarrow \infty$; this contradiction and the monotony of $k(L_n, B')$ show that $k(L_n, B') \rightarrow 0$. Lemma 3 shows that $k(L_n, B) \rightarrow 0$ also. A similar proof holds for $k(M_n, B)$; this can also be proved by using the remark before Lemma 4 and the fact that $k(L_n, B) \rightarrow 0$.

This theorem has as a corollary the result of [I]: *If $B = \mathfrak{P}^p(B_i)$, where $B_i = l^{p_i}$ or $L^{p_i}$, and if the numbers $p_i$ are not bounded away from*

[3] A. Sobczyk, *Projection of the space $m$ on its subspace $c_0$*, Bull. Amer. Math. Soc. vol. 47 (1941) pp. 938–947; the construction is given in the proof of Theorem 3.

1 *and* $\infty$, *then* $B$ *is not isomorphic to a uniformly convex space.*

It is not difficult to give a direct proof of Theorem 2 not using Theorem 1. I have also shown that if $B^*$ is uniformly convex, then $k(L_n, B)\rightarrow 0$ (as does $k(M_n, B)$); whether this condition is sufficient as well as necessary for isomorphism of $B$ or $B^*$ with a uniformly convex space is a question which I am, so far, unable to answer.

Some remarks may be made about the minimum values, $k(L_n)$ and $k(M_n)$, of $k(L_n, B)$ and $k(M_n, B)$ taken for $n$ fixed and $B$ varying over the spaces of dimension at least $n$. $k(M_n, l^p)=n^{-1/p}$ if $2\leqq p<\infty$ and $k(L_n, l^p)=n^{-1/p'}$ if $1\leqq p\leqq 2$ where $1/p'+1/p=1$. Hence $k(L_n)\leqq k(L_n, l^2)=n^{-1/2}$ and $k(M_n)\leqq k(M_n, l^2)=n^{-1/2}$ for all $n$. The plane with a regular hexagon for unit sphere is an example showing that $k(L_2)=k(M_2)\leqq 2/3$ $(<2^{-1/2})$. A tedious computation has shown that $2/3$ is precise; that is, that $k(L_2)=k(M_2)=2/3$. So far all my attempts to show $k(L_n)$ and $k(M_n)\geqq 1/n$ have failed for $n>2$.

The rest of this paper is devoted to extending the results of [II]. A normed vector space $T$ of real-valued functions $t=\{t_s\}$ on some set of indices $S$ will be called a *proper function space* if for every function $t=\{t_s\}$ in $T$ with $0\leqq t_s$ for all $s$ (a) for every real-valued function $\{t_s'\}$ with $0\leqq t_s'\leqq t_s$ for all $s$, the function $\{t_s'\}\in T$ and (b) $0\leqq\|\{t_s'\}\|\leqq\|\{t_s\}\|$. If $T$ is a proper function space and $B_s, s\in S$, are normed vector spaces, let $\mathfrak{P}_T\{B_s\}$ be the space of functions $b=\{b_s\}$ where $b_s\in B_s$ and the function $\{\|b_s\|\}\in T$; in $\mathfrak{P}_T\{B_s\}$, $\|b\|=\|\{b_s\}\|=\|\{\|b_s\|\}\|$. (In [II] $S$ was countable and only the special product spaces $\mathfrak{P}^p\{B_s\}=\mathfrak{P}_{l^p}\{B_s\}$ were used.)

THEOREM 3. *If $T$ is a proper function space, then $\mathfrak{P}_T\{B_s\}$ is uniformly convex if and only if $T$ is uniformly convex and the spaces $B_s$ have a common modulus of convexity.*

As the proof follows the lines of the proof of Theorem 3 of [II] except at one point it suffices to give the first half of the sufficiency proof; that is, the special case in which $\|b\|=\|b'\|=1$, $\|b-b'\|\geqq\epsilon$ and $\|b_s\|=\|b_s'\|$ for every $s$. Let $\beta_s=\|b_s\|$ and $\gamma_s=\|b_s-b_s'\|$; then for each $s$, $\|b_s+b_s'\|\leqq 2(1-\delta(\gamma_s/\beta_s))\beta_s$ where $\delta$ is a common modulus of convexity for all $B_s$. Hence

$$(1)\qquad \|b+b'\| = \|\{\|b_s+b_s'\|\}\|_T \leqq 2\|\{1-\delta(\gamma_s/\beta_s)\beta_s\}\|_T.$$

Clearly $\gamma_s\leqq 2\beta_s$ for all $s$; let $E$ be the set of all $s$ for which $\gamma_s/\beta_s>\epsilon/4$; then in $F$, the complement of $E$, $\beta_s\geqq 4\gamma_s/\epsilon$. If $\{t_s\}$ is any element of $T$, let $t_{sE}=t_s$ if $s\in E$, $t_{sE}=0$ if $s\notin E$; then

$$1 \geqq \|\{\beta_s\}\|_T \geqq \|\{\beta_{sF}\}\| \geqq \|\{4\gamma_{sF}/\epsilon\}\| = (4/\epsilon)\|\{\gamma_{sF}\}\|.$$

Hence $\|\{\gamma_{sF}\}\| \leqq \epsilon/4$ and

$$\|\{\gamma_{sE}\}\| = \|\{\gamma_s\} - \{\gamma_{sF}\}\| \geqq \|\{\gamma_s\}\| - \|\{\gamma_{sF}\}\| \geqq 3\epsilon/4.$$

Hence $\|\{\beta_{sE}\}\| \geqq \|\{\gamma_{sE}\}\|/2 \geqq 3\epsilon/8$.

Now let $t = \{\beta_{sF}\}$, $t' = \{\beta_{sE}\}$ and $t'' = (1-2\delta(\epsilon/4))t'$; then $\|t+t''\| \leqq \|t+t'\| = 1$ and $\|t+t'-(t+t'')\| = \|t'-t''\| = 2\delta(\epsilon/4)\|t'\| \geqq 3\delta(\epsilon/4)\epsilon/4$. Call this last quantity $\alpha(\epsilon)$; then

$$(2) \quad \|(1-\delta(\epsilon/4))t' + t\| = (1/2)\|t + t' + t + t''\| \leqq 1 - \delta_1(\alpha(\epsilon))$$

where $\delta_1$ is the function which exists in $T$ by Lemma 1. By (1) and (2)

$$\|b + b'\| \leqq \|\{(1 - \delta(\gamma_s/\beta_s))\beta_{sE}\} + \{\beta_{sF}\}\| \leqq \|(1 - \delta(\epsilon/4))t' + t\|$$
$$\leqq 1 - \delta_1(\alpha(\epsilon)) \equiv 1 - \delta_2(\epsilon).$$

The remainder of the proof is exactly that given in [II] (beginning with line 4 on p. 506); it shows that a suitable value of $\delta_3$ in $\mathcal{P}_T\{B_s\}$ is given if $\delta_3(\epsilon) = \delta_1(\eta)$ where $\eta$ is so chosen that $\eta/2 + \delta_1(\eta) < \delta_2(\epsilon)$. Since $\delta_3$ depends only on the moduli of convexity in $T$ and all $B_s$, we have the following result, more general than Corollary 1 of [II].

COROLLARY. *If $\{T\}$ is a collection of proper function spaces, if $\{B\}$ is a collection of normed vector spaces, and if all these spaces have a common modulus of convexity, then all the spaces $\mathcal{P}_T\{B_s\}$ with $T$ in $\{T\}$ and all $B_s$ in $\{B\}$ have a common modulus of convexity.*

Some extensions of Theorem 3 may be made; for instance, it is clear that the condition (a) on a proper function space is imposed to make sure that such functions as $\{\|b_s+b_s'\|\}$ are in $T$. For example, if $S$ is a space in which a measure is defined and all $B_s$ are the same space $B_0$, it suffices to take $T=L_S^p$, $1<p<\infty$ and to consider only Bochner measurable functions[4] $\{b_s\}$ for which $\{\|b_s\|\} \in T$. In this case all the functions constructed are again in $T$ so the proof can be carried through showing directly that $L^p(B_0)$ is uniformly convex if $1<p<\infty$ and $B_0$ is uniformly convex. In fact, if the norm in $T$ satisfies (b) and if it is assumed only that every measurable real-valued function dominated by a function in $T$ is again in $T$, the proof can be carried through for the space of Bochner measurable functions from $S$ into $B$ for which $\{\|b_s\|\} \in T$.

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[4] S. Bochner, *Integration von Funktionen, deren Werte die Elemente eines Vektorraumes sind*, Fund. Math. vol. 20 (1933) pp. 262–276.