depend on the remainder in the division of t^k by $\psi_m(t)-F_m$ and $\psi_m(t)$ respectively. (Cf. H. L. Lee, Duke Math. J. vol. 9 (1943) pp. 277–292.) (Received July 19, 1943.)

202. W. V. Parker: *Limits to the characteristic roots of a matrix.*

Let $A=(a_{ij})$ be a square matrix of order n with elements in the field of complex numbers; and define $S_i=\sum_{j=1}^n|a_{ij}|$, $T_j=\sum_{i=1}^n|a_{ij}|$, $U_i=2|a_{ii}|-S_i$, and $V_j=2|a_{jj}|-T_j$. Let S, T be the greatest of the S_i, T_j, respectively; and let U, V be the least of the U_i, V_j, respectively. It is shown that the absolute value of each characteristic root of A is not less than the greater of the numbers U and V and is not greater than the smaller of the numbers S and T. Similar bounds are also found for the real and imaginary parts of the characteristic roots. (Received July 23, 1943.)

203. H. E. Salzer: *Table of first two hundred squares expressed as a sum of four tetrahedral numbers.*

The following empirical theorem is conjectured: Every square integer is expressible as the sum of four positive (including zero) tetrahedral numbers $(n^3-n)/6$. It has been verified by a table prepared for the first 200 squares. This empirical theorem is a partial improvement of the statement that five non-negative tetrahedrals suffice for any integer. (See F. Pollock, Proc. Roy. Soc. London Ser. A. vol. 5 (1850).) (Received June 4, 1943.)

ANALYSIS

204. R. H. Cameron and W. T. Martin: *Transformations of Wiener integrals under translations.*

Let $F[y]$ be a functional defined and Wiener summable over the space C consisting of all functions $x(t)$ continuous in $0\leqq t\leqq 1$ and vanishing at $t=0$. In addition, let F be continuous and let it be bounded over every bounded set $x(\cdot)$ of C. (F is called continuous if $F[y^{(n)}]\to F[y^{(0)}]$ whenever $y^{(n)}(t)\to y^{(0)}(t)$ uniformly in $0\leqq t\leqq 1$, and F is bounded over every bounded set $x(\cdot)$ of C if for every positive constant B there exists a constant $K=K_B$ such that $|F[y]|\leqq K$ for all $y(\cdot)$ of C for which $|y(t)|\leqq B$, $0\leqq t\leqq 1$.) Under these conditions on the functional F the authors obtain a transformation formula for Wiener integrals under translations of the form $y(t)=x(t)+x_0(t)$ where $x_0(t)$ is a given function of C with a first derivative $x_0'(t)$ of bounded variation in $0\leqq t\leqq 1$. The transformation formula is $\int_C^w F[y]d_wy=\int_C^w F[x+x_0]\exp\{-\int_0^1[x_0'(t)]^2dt-2\int_0^1x_0'(t)dx(t)\}d_wx$. The formula forms a basis for the calculation of various types of Wiener integrals. (Received July 30, 1943.)

205. M. M. Day: *Uniform convexity.* IV.

In this paper relationships between uniform convexity, factor spaces, and conjugate spaces are discussed. Theorem 1: A normed vector space B is uniformly convex if and only if all the two dimensional factor spaces of B are uniformly convex with a common modulus of convexity. The concept of uniform flattening is suggested by a description of a "sharp edge" on the unit sphere in terms of the norm of the space. It is shown [Theorem 2] that this is dual to uniform convexity; that is, $B[B^*]$ is uniformly flattened if and only if $B^*[B]$ is uniformly convex. It follows that a complete uniformly flattened B is reflexive. The proof of Theorem 2 uses a computation for two dimensional spaces of the flattening in B in terms of the convexity in B^* and vice versa; Theorem 1 is then applied to complete the proof. (Received June 16, 1943.)

206. G. M. Ewing: *Existence theorems for multiple integral variation problems.* Preliminary report.

This is a sequel to a previous paper by the author (abstract 49-1-30) shortly to appear in the Duke Math. J. The methods used there for the single integral case are extended to obtain existence theorems under fairly general hypotheses for the problem of minimizing a certain k-tuple integral, $\int f(x, X)du$, on a class of continuous k-surfaces in euclidean n-space. Metric methods are used to obtain a convergent minimizing sequence but the final theorems closely resemble the Tonelli pattern. (Received July 23, 1943.)

207. Tomlinson Fort: *The weighted vibrating string and its limit.*

In this paper the problem of the vibrating string weighted with discrete particles is studied following lectures given by Bôcher years ago at Harvard. In addition to an outline of the work of Bôcher, an interesting special case is studied. A careful passage to the limit is carried through, thus obtaining the solution of the problem of the vibrating nonhomogeneous string in terms of Sturm-Liouville functions. It is believed that not only is the method new but the final form somewhat different from any previous presentation. Incidentally, under proper restrictions, the solution of a differential equation system is proved the limit of the solution of a recurrence system. The method of proof is the method of successive approximations. (Received August 4, 1943.)

208. Szolem Mandelbrojt: *Quasi-analyticity and analytic continuation. A general principle.*

The author proves a theorem which leads to results in different branches of the theory of functions. Results on quasi-analyticity on one hand, and results on the distribution of the singularities of a Dirichlet series and the distribution of the values taken by its analytic continuation, on the other hand, may be considered as particular cases of this theorem. (Received July 29, 1943.)

209. D. S. Miller: *On Carathéodory and Gillespie linear measure.*

In this paper the relation between the Carathéodory linear measure $L(A)$ and the Gillespie linear measure $G(A)$ of a plane measurable point set will be considered. The inequality $L(A) \leqq G(A) \leqq (\pi/2)L(A)$ will be shown to be the best possible by presenting sets E and F such that $L(E) = G(E)$ and $G(F) = (\pi/2)L(F)$. (Received August 2, 1943.)

210. P. C. Rosenbloom: *Sequences of polynomials, especially sections of power series.*

The domain of boundedness B of a sequence of polynomials is defined as the set of points z such that the sequence is uniformly bounded in some neighborhood of z. Relations are obtained between B, the gap-like structure of the polynomials, and the distribution of their zeros. The connection between the moduli of the zeros and the coefficients is studied in detail. It is proved that under certain rather general conditions the arguments of the zeros are everywhere dense. If $f(z)=a_0+a_1z+\cdots$, then the order of magnitude of the zeros of $s_n(z)$ is roughly $r_n=|a_n|^{-1/n}$ for a certain subsequence of n's. The arguments of the zeros are everywhere dense except if $f(z)$ is an entire function of zero order. Fairly complete results for the case of a finite or zero radius of convergence, or of an entire function of infinite order, are obtained. For entire functions of finite positive order the author obtains essentially all the results announced by Carlson (C. R. Acad. Sci. Paris vol. 179 (1924) pp. 1583–1585), whose proofs have never been published so far as is known to the author. A connection between the zeros of $s_n(z)$ and the conformal mapping $w=z^{-n}f(r_nz)$ is also obtained. These results are applied to special power series. (Received July 31, 1943.)

211. Raphael Salem: *A singularity of the Fourier series of continuous functions.*

The following theorems are proved: I. There exists a continuous function $f(x)$ of period 2π whose Fourier series converges everywhere uniformly, and such that the Fourier series of $f^2(x)$ diverges at a point. II. There exists a continuous function $f(x)$ of period 2π whose Fourier series converges everywhere uniformly, and such that the Fourier series of $f^2(x)$ diverges at an everywhere dense set of points having the power of the continuum. (Received July 29, 1943.)

212. A. R. Schweitzer: *On functional equations with solutions containing arbitrary functions.* I.

The author outlines a heuristic method of detecting functional equations (mainly of the iterative compositional type) whose solutions involve arbitrary functions, by means of generalization of linear functions of variables which are solutions and which contain arbitrary constants as coefficients. It is assumed hypothetically that if a functional equation has a solution of the indicated type then the solution persists if that part of the linear function containing arbitrary constants is suitably replaced by arbitrary functions of the corresponding variables. The method indicated is heuristic in the sense defined in the author's article, *Revue de mêtaphysique et de morale*, 1914 (Le principe de continuation). If the method does not succeed, other functional equations frequently result whose solutions approach the solution of the original equation. The preceding effects a gradual transition from functional equations previously solved by the author by differentiation (such as certain of his quasi-transitive equations) to equations at least not readily solved in the latter manner. The preceding type of solutions is contrasted with solutions of equations obtained analogously by generalizing linear functions of particular solutions containing arbitrary constants as coefficients. Reference is made to papers previously reported in this Bulletin (vol. 23 (1917) pp. 300, 393; vol. 24 (1918) p. 279). (Received July 31, 1943.)

213. A. R. Schweitzer: *On functional equations with solutions containing arbitrary functions.* II.

The author applies the method of the preceding paper to the solution of functional equations previously defined by him and equations relevant to number systems (such as distributive and generalized associative equations) and in suitable instances, their formal inverses relatively to equations of the type: $\phi[f(x, y_1, y_2, \cdots, y_n), y_1, y_2, \cdots,$

$y_n] = x$. Additional hypotheses are of the type: If $f(x_1, x_2, \cdots, x_{n+1})$ is a solution, then $\psi^{-1}f[\psi(x_1), \psi(x_2), \cdots, \psi(x_{n+1})]$ is also a solution where $\psi(x)$ is an arbitrary function with an inverse. Example: $\zeta(u, y_1, y_2, \cdots, y_n) = \tau(v_1, v_2, \cdots, v_{n+1})$ where $u = \tau(x_1, x_2, \cdots, x_{n+1})$ and $v_i = \zeta(x_i, y_1, y_2, \cdots, y_n)$. Solution: $\zeta(x_1, x_2, \cdots, x_{n+1}) = x_1 \cdot \alpha(x_2, x_3, \cdots, x_{n+1})$ and $\tau(x_1, x_2, \cdots, x_{n+1}) = \beta(w_1, w_2, \cdots, w_n) \cdot x_{n+1}$ where $w_i = x_i/x_{n+1}$ and α and β are arbitrary functions. The preceding equation is abstractly self-inverse. Other examples with solutions are given. The solutions indicated are of course not exhaustive. Application is made in suitable instances to the domain of abstract groups; for instance, for $n = 1$ the preceding equation has the solution: $\zeta(x_1, x_2) = x_1 \cdot x_2^p$, $\tau(x_1, x_2) = (x_1 \cdot x_2^{-1})^q \cdot x_2$ where p and q are arbitrary integers; and analogously for $n > 1$. Reference is made to papers previously reported in this Bulletin (vol. 25 (1919) pp. 250, 257). (Received July 31, 1943.)

214. Andrew Sobczyk: *On the extension of linear transformations.*

By the Hahn-Banach theorem, any linear transformation T on a subspace X of a normed linear space Z, having a one-dimensional range, may be extended to the whole space with preservation of the norm. It is shown that the same is true for any linear transformation, provided that the range Y may be enlarged to a normed linear space $W \supset Y$. Specifically, a norm-preserving extension is always possible with range a subspace of the quotient space Z/X', where $X' \subset X$ is the zero subspace of T, and the norm in Z/X' is an extension of the norm on $Y \cong X/X'$ with respect to the Banach norm $\inf_{x' \epsilon X'} \|z + X'\|$. Results are obtained on the question of when the existence of an extension requires the existence of a projection. A classification of linear transformations into four fundamental types is given, with examples showing the existence of each type. (Received July 29, 1943.)

215. Otto Szász: *On uniform convergence of Fourier series.*

The main object of this paper is to establish conditions for uniform convergence of a Fourier series at a given point. A necessary condition is that the function be continuous at that point; the additional condition restricts the coefficients to a certain type of slow oscillation, such as were used in Tauberian problems. The result yields Gibbs' phenomenon for such series at points of discontinuity. A theorem of a converse nature is: If $f(\theta) \sim \sum b_n \sin n\theta$ is continuous at $\theta = 0$, and if for some positive constants K and C, $0 \leqq (n+1)b_{n+1} + K \leqq (1 + C/n)(nb_n + K)$, $n \geqq 1$; then $nb_n \to 0$. (Received July 26, 1943.)

216. H. S. Wall and Marion D. Wetzel: *Positive definite forms and convergence theorems for continued fractions.*

The form (1) $\sum_{p=1}^{n} (\beta_p + y)x_p^2 - 2\sum_{p=1}^{n-1} \alpha_p x_p x_{p+1}$ is positive definite for all $y > 0$ if and only if (2) $\beta_p \geqq 0$, $\alpha_p^2 = \beta_p \beta_{p+1} (1 - g_{p-1})g_p$, $0 \leqq g_r \leqq 1$. The J-fraction (3) $1/(b_1 + z) - a_1^2/(b_2 + z) - a_2^2/(b_3 + z) - \cdots$ is called positive definite if $\beta_p = \Im(b_p)$, $\alpha_p^2 = \Im(a_p)$ satisfy (2). There exists a nest of circles $K_p(z) \supset K_{p+1}(z)$ such that the pth approximant of (3) is upon $K_p(z)$, $(\Im(z) > 0)$. The distinction between the two cases $r_p(z) \to 0$, $r_p(z) \to r(z) > 0$ ($r_p(z)$ = radius of $K_p(z)$) is invariant under change in the value of z in $\Im(z) > 0$. If $\lim \inf |a_p|$ is finite, then $r_p(z) \to 0$. This, along with (2), leads to the theorem that the continued fraction (4) $1/1 + c_1/1 + c_2/1 + \cdots$ converges if $\lim \inf |c_p| < \infty$ and $|c_p| - \Re(c_p) \cos(\phi_p + \phi_{p+1}) + \Im(c_p) \sin(\phi_p + \phi_{p+1}) \leqq [2 \cos \phi_p \cos \phi_{p+1}(1 - g_{p-1})g_p]/$

$[(1+y \sec \phi_p)(1+y \sec \phi_{p+1})]$, $y>0$, $-(\pi/2)<\phi_p<+(\pi/2)$, $0 \leqq g_p \leqq 1$. Also, (4) converges if $\liminf |c_p| < \infty$ and $\sum [|c_p| - \Re(c_p)] < 2$. (Received July 2, 1943.)

217. F. T. Wang: *On Riesz summability of Fourier series.* III.

Denote by $\phi_\alpha(t)$ the fractional integral of order α of an even integrable function $\phi(t)$, periodic, with period 2π. The following two results are proved: (i) If $\phi_\alpha(t) = o(t^\gamma)$, $\gamma>\alpha>0$, then the Fourier series of $\phi(t)$ is summable $(R, e^{n^{1-\alpha/\gamma}}, \gamma+\delta)$, $\delta>0$, at $t=0$. (ii) If $\phi_\alpha(t) = o(t^\alpha(\log t)^{-1})$, $\alpha>0$, then the Fourier series of $\phi(t)$ is summable $(R, n^{(\log n)^{1/\alpha}}, \alpha+1)$ at $t=0$. (Received June 11, 1943.)

Applied Mathematics

218. Stefan Bergman: *Solutions of linear partial differential equations of fourth order.*

Generalizing the known formula $u = \mathrm{Re}[f_1(z) + \bar{z}f_2(z)]$, $z = x+iy$, $\bar{z} = x-iy$, for the biharmonic functions (that is, functions u for which $\Delta\Delta u = 16u_{zz\bar{z}\bar{z}} = 0$), the author proves that for every equation $L(u) = U_{zz\bar{z}\bar{z}} + a_1u_{zz} + a_2u_{z\bar{z}} + a_3u_{\bar{z}\bar{z}} + a_4u_z + a_5u_{\bar{z}} + a_6u = 0$ where $a_\nu = a_\nu(z, \bar{z})$ are analytic functions of z and $\bar{z}$ there exist two functions of z, $\bar{z}$ and a real variable t, $\mathrm{E}_k(z, \bar{z}, t)$, $k=1, 2$, such that every solution of $L(u)=0$ which is regular in a star domain $\mathfrak{D}$ can be represented in $\mathfrak{D}$ in the form $u(\bar{z}) = \mathrm{Re}\{\int_{-1}^{+1}[\sum_{k=1}^{2} \mathrm{E}_k(z, \bar{z}, t) f_k(z(1-t^2)/2)dt/(1-t^2)^{1/2}]\}$. The methods and results of the paper *Linear operators in the theory of partial differential equations* (Trans. Amer. Math. Soc. vol. 53 pp. 130–155) can be applied to the functions u, satisfying $L(u)=0$. (Received July 30, 1943.)

219. G. E. Forsythe: *Note on equivalent potential temperature.*

Let a parcel of moist air be saturated with $w_s = w_s(p, T)$ tons of water vapor per ton of dry air. Let the following quantities be measured in meter-ton-second-absolute degree mechanical units: T=temperature; p=total air pressure; $L = L(T)$=latent heat of evaporation of water; $c_p = c_p(T)$ $[c_v = c_v(T)]$=specific heat of dry air at constant pressure [volume]; $e_s = e_s(T)$=vapor pressure of saturated water vapor; $p_d = p - e_s$; $k = (c_p - c_v)/c_p$; $\theta_d = T(100/p_d)^k$. Let $\lambda = \lambda(p, T) = Lw_s/(c_pT)$. Rossby (Massachusetts Institute of Technology Meteorological Papers, vol. I, no. 3 (1932)) defines the equivalent potential temperature θ_e by the relation $\theta_e = \theta_d \exp \lambda$. He asserts without proof that, as $p \to 0$ in a process for which θ_e is constant, one has $\theta_d \to \theta_e$. The present note uses elementary estimates to prove Rossby's assertion: (i) under the oversimplifying assumption that L and c_p are bounded away from 0, as $T \to 0$; (ii) under weaker but physically artificial assumptions about L and c_p. The note includes a further discussion of the important meteorological quantity λ. (Received July 26, 1943.)

220. S. H. Gould: *The Rayleigh-Ritz method for higher eigenvalues.*

An elementary proof is given of the following theorem, fundamental in applications of the Rayleigh-Ritz method for calculating the eigenvalues of a given variational eigenvalue problem: the n roots, arranged in order of magnitude, of the determinantal equation obtained by using n coordinate functions are upper bounds respectively for the first n eigenvalues of the original problem. (Received July 14, 1943.)