THE DISTRIBUTION OF INTEGERS REPRESENTED BY BINARY QUADRATIC FORMS

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R. D. James[1] has proved the following theorem:

THEOREM 1. *Let $B(x)$ denote the number of positive integers $m \leqq x$ which can be represented by positive, primitive, binary quadratic forms of a given negative discriminant d, but are prime to d. Then*

$$B(x) = bx/(\log x)^{1/2} + O(x/\log x), \tag{1}$$

where b is the positive constant given by

$$\pi b^2 = \prod_q (1 - 1/q^2)^{-1} \prod_{p \mid d} (1 - 1/p) \sum_{n=1}^{\infty} (d \mid n) n^{-1}. \tag{2}$$

Here q runs over all primes such that $(d \mid q) = -1$; p denotes any prime greater than or equal to 2; and $(d \mid n)$ is the Kronecker symbol.

We shall deduce from his result an asymptotic formula with the restriction that m be prime to d removed.

First, let p be a prime dividing d but not satisfying

$$p > 2 \text{ and } p^2 \mid d, \quad \text{or} \quad p = 2 \text{ and } d \equiv 0 \text{ or } 4 \pmod{16}. \tag{3}$$

Then pn is represented by p. p. b. q. forms of discriminant d if and only if n is likewise represented.[2] Hence if $p^r \leqq (\log x)^{1/2}$ then the number of represented integers less than or equal to x of the form $p^r m$ with m prime to d, is

$$\frac{bx/p^r}{(\log x/p^r)^{1/2}} + O\left(\frac{x/p^r}{(\log x/p^r)}\right) = \frac{bx/p^r}{(\log x)^{1/2}} + O\left(\frac{x/p^r}{\log x}\right),$$

since $(\log xp^{-r})^{-1/2} - (\log x)^{-1/2} = O((\log p^r)/(\log x)^{3/2})$. Also, if $p^s > (\log x)^{1/2}$,

$$(bx/(\log x)^{1/2})(1/p^s + 1/p^{s+1} + \cdots) = O(x/(\log x)).$$

Hence the number of positive integers less than or equal to x, represented by p.p.b.q. forms of discriminant d, and prime to d except that they need not be prime to p, is given by

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[1] R. D. James, Amer. J. Math. vol. 60 (1938) pp. 737–744.

[2] G. Pall, Math. Zeit. vol. 36 (1933) pp. 321–343, p. 331.

$$\frac{bx}{(\log x)^{1/2}}(1 + 1/p + \cdots + 1/p^r + \cdots) + O(x/\log x).$$

Thus each prime p dividing d but not satisfying (3) can be removed from the set of primes excluded in m, by replacing b by $b/(1-1/p)$. We recall that[3]

$$\sum (d \mid n)n^{-1} = 2\pi h(d)/(w(-d)^{1/2}), \tag{4}$$

where $h(d)$ is the number of p.p.b.q. classes of discriminant d, and $w=2$, except that $w=4$ if $d=-4$, $w=6$ if $d=-3$. Hence the number of positive integers less than or equal to x which can be represented by p.p.b.q. forms of discriminant d, and which have no prime factor p satisfying (3), is

$$b_0x/(\log x)^{1/2} + O(x/\log x),$$

where

$$\begin{aligned} b_0^2 = (2h(d)/w(-d)^{1/2}) \prod_q (1 - 1/q^2)^{-1} \\ \prod_{p \text{ sat. } (3)} (1 - 1/p) \prod_{p|d, \text{ not } (3)} (1 - 1/p)^{-1}. \end{aligned} \tag{5}$$

Second let p satisfy (3). Then[4] pm is not represented by p.p.b.q. forms of discriminant d if $p \nmid m$; and p^2n is represented by such forms if and only if n is represented by p.p.b.q. forms of discriminant $d_1 = d/p^2$. If b_1 is the value of b_0 corresponding to d_1, then from (5),

$$\begin{aligned} b_1 &= b_0 && \text{if } d_1/p^2 \text{ is an integer} \equiv 0 \text{ or } 1 \bmod 4, \\ &= b_0(1 - p^{-2})^{-1} && \text{if } (d_1 \mid p) = -1, \\ &= b_0(1 - p^{-1})^{-1} && \text{otherwise.} \end{aligned} \tag{6}$$

Hence if $d = p^{2k}d'$, where $k \geqq 1$ and $p^2 \nmid d'$, we can remove p from the set of excluded primes by replacing $b_0x/(\log x)^{1/2}$ by $(b_0x/(\log x)^{1/2})$ $(1+p^{-2}+ \cdots +p^{-2k+2}+p^{-2k}\lambda)$, where $\lambda = 1/(1-1/p)$ if $(d' \mid p) \neq -1$, $\lambda = 1/(1-1/p^2)$ if $(d' \mid p) = -1$; hence by multiplying b_0 by

$$\frac{1 + p^{-2k-1}}{1 - p^{-2}} \text{ if } (d' \mid p) \neq -1, \qquad \frac{1}{1 - p^{-2}} \text{ if } (d' \mid p) = -1.$$

Thus we have finally this theorem:

THEOREM 2. *The number $C(x)$ of positive integers $n \leqq x$ which can be*

[3] See, for example, Landau, *Vorlesungen über Zahlentheorie*, vol. 2, p. 152.

[4] G. Pall, Amer. J. Math. vol. 57 (1935) pp. 789–799, formula (9).

represented by positive, primitive, binary quadratic forms of a given negative discriminant d is given by

$$cx/(\log x)^{1/2} + O(x/\log x),$$

where c is the positive constant defined by

$$c = b_0 \prod_{p \text{ sat. } (3)} (1 - p^{-2})^{-1} \cdot \prod (1 + p^{-2k-1}), \tag{7}$$

where in the last product $d = p^{2k}d'$, where $p^2 \nmid d'$, $k \geqq 1$, and $(d' | p) \neq -1$.

Example. If $d = -3$,

$$c^2 = b_0^2 = 3^{-1} \cdot 3^{-1/2} \cdot \alpha \cdot (3/2), \qquad \alpha = \prod_{q}^{q \equiv 2(3)} (1 - q^{-2})^{-1},$$

$$= \alpha/(2(3)^{1/2}), \qquad c = .64 \text{ approximately.}$$

If $d = -12$, $b_0^2 = (1/(12)^{1/2}) \prod'_{q(\neq 2)} (1 - q^{-2})^{-1} \cdot (1/2) \cdot (3/2) = 9\alpha/(32(3)^{1/2})$; and by (7), $c^2 = b_0^2(16/9) = \alpha/(2(3)^{1/2})$. Hence c is the same for $d = -12$ as for $d = -3$. This agrees with the fact that $x^2 + 3y^2$ represents exactly the same numbers as $x^2 + xy + y^2$.

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