A NOTE ON APPROXIMATION BY RATIONAL FUNCTIONS

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The theory of the approximation by rational functions on point sets E of the z-plane ($z=x+iy$) has been summarized by J. L. Walsh[1] who himself has proved a great number of important theorems some of which are fundamental. The results concern both the case when E is bounded and when E extends to infinity.

In the present note a L_p-theory ($0<p<\infty$) will be given for the following point sets extending to infinity:

A. The real axis $-\infty<x<\infty$, $y=0$.

B. The half-plane $-\infty<x<\infty$, $0<y<\infty$.

The only poles of the approximating functions are to lie at preassigned points whose number will be required to be as small as possible.[2] We shall make use of the theory of the class $\mathfrak{H}_p$ the fundamental results of which are due to E. Hille and J. D. Tamarkin;[3] $\mathfrak{H}_p$ is the set of functions $F(z)$ which, for $0<y<\infty$, are regular and satisfy the inequality

$$\int_{-\infty}^{\infty} |F(x+iy)|^p dx \leqq M^p \quad \text{or} \quad |F(z)| \leqq M$$

for $0<p<\infty$ or $p=\infty$, respectively, where M depends on F and p only. By $|f(x+iy)|_p$ we denote

$$\left(\int_{-\infty}^{\infty} |f(x+iy)|^p dx\right)^{1/p} \quad \text{or} \quad \operatorname*{ess.\,u.b.}_{-\infty<x<\infty} |f(x+iy)|$$

for $0<p<\infty$ or $p=\infty$, respectively, and by α and β two arbitrarily fixed points in the upper or lower half-plane, respectively. We obtain the following results:[4]

THEOREM 1. *Let $0<p<\infty$ and $F(t)\in L_p(-\infty,\infty)$, let c be an integer greater than p^{-1} and $r_k(z)=(\alpha-z)^k(z-\beta)^{-c-k}$ [$k=0, \pm 1, \pm 2, \cdots$].*

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[1] *Interpolation and approximation by rational functions in the complex domain,* Amer. Math. Soc. Colloquium Publications, vol. 20, 1935.

[2] Compare Walsh, loc. cit., for example, approximation by polynomials.

[3] Fund. Math. vol. 25 (1935) pp. 329–352, $1\leqq p<\infty$. For $0<p<1$ see T. Kawata, Jap. J. Math. vol. 13 (1936) pp. 421–430.

[4] The case $p=\infty$ of each of the results is a special case of Theorem 16, J. L. Walsh, chap. 2.

Then there are finite linear combinations $s_n(z)$ *of the* $r_k(z)$ *such that*

$$| F(t) - s_n(t)|_p = \left(\int_{-\infty}^{\infty} | F(t) - s_n(t) |^p dt\right)^{1/p} \to 0 \qquad \text{as } n \to \infty.$$

THEOREM 2. (a) *Let* $0<p<\infty$ *and* $F(t)\in L_p(-\infty, \infty)$. *A necessary and sufficient condition for the existence of rational functions* $s_n(z)$ *such that their only poles lie in a single point of the lower half-plane and that* $|F(t)-s_n(t)|_p\to 0$ *as* $n\to\infty$ *is that* $F(t)$ *is equivalent to the limit-function of an element* $F(z)$ *of* $\mathfrak{H}_p$.

(b) *When the latter condition is satisfied then there are rational functions* $s_n(z)$, *with their only poles at* $z=\beta$, *such that, uniformly in the half-plane* $0<y<\infty$,

$$| F(x + iy) - s_n(x + iy) |_p \to 0 \qquad \text{as } n \to \infty.$$

By a well known result[5] concerning $\mathfrak{H}_p$, 2(b) is a consequence of 2(a).

We start with giving explicit approximating functions in some special cases of problem (A), taking $\beta=\bar{\alpha}$.

THEOREM 1′. *Let* $F(t)\in L_1(-\infty, \infty)$ *or* $F(t)\in L_2(-\infty, \infty)$, *or let* $F(t)$ *be continuous everywhere, including infinity.*[6] *Let* $c=2, 1, 0$ *for* $p=1, 2, \infty$, *respectively, and let*

$$s_n(z) = \sum_{k=-n}^{n} a_k \frac{(\alpha - z)^k}{(z - \bar{\alpha})^{k+c}}, \qquad a_k = \frac{i(\alpha - \bar{\alpha})}{2\pi}\int_{-\infty}^{\infty} F(t) \frac{(t - \bar{\alpha})^{k+c-1}}{(\alpha - t)^{k+1}} dt.$$

Then

$$\left| F(t) - \frac{1}{N+1} \sum_{n=0}^{N} s_n(t) \right|_1 \quad or \quad | F(t) - s_N(t) |_2 \quad or$$

$$\left| F(t) - \frac{1}{N+1} \sum_{n=0}^{N} s_n(t) \right|_\infty = \operatorname*{u.b.}_{-\infty<t<\infty} \left| F(t) - \frac{1}{N+1} \sum_{n=0}^{N} s_n(t) \right|,$$

respectively, tends to zero as $N\to\infty$. *When* $F(t)$ *is continuous everywhere, including infinity, and of bounded variation in* $(-\infty, \infty)$ *then the* $s_n(t)$ *converge to* $F(t)$ *uniformly in* $(-\infty, \infty)$.

It will suffice to take $\alpha=i$, the general case being deduced from this one by the substitution $t=\Re(\alpha)+t'\Im(\alpha)$. Let $F(t)\in L_2(-\infty, \infty)$, $t=\tan(1/2)\vartheta$ $[-\pi\leqq\vartheta\leqq\pi]$, and $f(\vartheta)=2(1+e^{i\vartheta})^{-1}F(\tan\vartheta/2)$. Then $F(t)\in L_2(-\infty, \infty)$ implies that $f(\vartheta)\in L_2(-\pi, \pi)$, and vice versa. Now

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[5] Since $s_n(t)\in L_p$, we have $s_n(z)\in\mathfrak{H}_p$, $F(z)-s_n(z)\in\mathfrak{H}_p$, and we can apply the Hille-Tamarkin Theorem 2.1 (iii), part 2, loc. cit.

[6] A function $F(t)$ is said to be continuous at infinity when its limits, as $t\to\pm\infty$, both exist and are finite and equal.

the Fourier series $\sum b_n e^{in\vartheta}$, belonging to $f(\vartheta)$, converges to $f(\vartheta)$ in the mean square over $(-\pi, \pi)$. We have $e^{i\vartheta}=(i-t)(i+t)^{-1}$, $(1/2)(1+e^{i\vartheta}) = i(i+t)^{-1}$; taking $a_n = ib_n$, we arrive finally at the required result. In a similar way we prove the remaining assertions of the theorem. We note[7] that the sequence $\{(2i\pi)^{-1/2}(\alpha-\bar{\alpha})^{1/2}(\alpha-t)^n(t-\bar{\alpha})^{-n-1}\}$ $[n=0, \pm 1, \pm 2, \cdots]$ is a complete orthogonal and normal system with respect to $L_p(-\infty, \infty)$ $[1<p<\infty]$.

To prove Theorem 1, we have to show that, given $\epsilon>0$, there is a finite linear combination $s_n(z)$ of the $r_k(z)$ such that $|F(t)-s_n(t)|_p<\epsilon$. We can find a positive number b and a function $f(t)$ such that $f(t)$ is zero for $|t| \geqq b$ and continuous for $-b \leqq t \leqq b$, and that

$$\int_{-\infty}^{\infty} |F(t) - f(t)|^p dt \leqq \delta, \qquad \delta = \begin{cases} (\epsilon/2)^p & \text{for } p > 1 \\ (1/2)\epsilon^p & \text{for } p \leqq 1. \end{cases}$$

The function $g(t)=(t-\beta)^c f(t)$ is continuous everywhere, including infinity. From results of Walsh[8] we deduce the existence of functions

$$\sigma_n(z) = \sum_{k=-n}^{n} a_{k,n}\left(\frac{\alpha - z}{z - \beta}\right)^k, \qquad n = 0, 1, 2, \cdots,$$

$|g(t)-\sigma_n(t)|_\infty \to 0$ as $n\to\infty$. Taking $s_n(z)=(z-\beta)^{-c}\sigma_n(z)$, we have

$$|f(t) - s_n(t)|_p^p = \left|\frac{g(t) - \sigma_n(t)}{(t-\beta)^c}\right|_p^p \leqq |g(t) - \sigma_n(t)|_\infty^p \int_{-\infty}^{\infty} \frac{dt}{|t-\beta|^{cp}}.$$

The right side tends to zero as $n\to\infty$. Therefore, for some n, we have $|f(t)-s_n(t)|_p^p<\delta$, $|F(t)-s_n(t)|_p^p<\epsilon^p$ which completes the proof.

To prove Theorem 2(a), we need some lemmas.

LEMMA 1.[9] *Let $\varphi(w)$ belong to the Riesz class H_p $[0<p<\infty]$, that is to say, let $\varphi(w)$ be regular for $|w|<1$ and satisfy the inequality*

$$\|\varphi(re^{i\vartheta})\|_p = \left(\int_{-\pi}^{\pi} |\varphi(re^{i\vartheta})|^p d\vartheta\right)^{1/p} \leqq M, \qquad 0 < r < 1,$$

where M is independent of r.[10] *Then there are polynomials $P_n(w)$ $[n=1, 2, \cdots]$ such that $\|\varphi(re^{i\vartheta})-P_n(re^{i\vartheta})\|_p \to 0$ as $n\to\infty$, uniformly for $0<r\leqq 1$.*

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[7] Cf. H. Kober, a forthcoming paper in Quart. J. Math. Oxford Ser. 1943.

[8] Walsh, loc. cit. chap. 2, Theorem 16. It can also be deduced from Theorem 1′ of this paper.

[9] For $p=\infty$ the result holds if and only if $\phi(e^{i\vartheta})$ is continuous for $-\pi\leqq\vartheta\leqq\pi$. Cf. Walsh, loc. cit., and Trans. Amer. Math. Soc. vol. 26 (1924) pp. 155–170.

[10] F. Riesz, Math. Zeit. vol. 18 (1923) pp. 87–95.

By well known properties of the class H_p, it will suffice to take $r=1$. Let $\varphi(w)=\sum a_n w^n$. Since, for any fixed R $[0<R<1]$ and uniformly with respect to ϑ $[-\pi\leqq\vartheta\leqq\pi]$, the series $\sum a_n R^n e^{in\vartheta}$ converges to $\varphi(Re^{i\vartheta})$, the result can be deduced by means of the well known equation $\|\varphi(e^{i\vartheta})-\varphi(re^{i\vartheta})\|_p\to 0$ $[r\to 1]$.

LEMMA 2. *Let $w=(i-z)(i+z)^{-1}$. The function $F(z)$ belongs to $\mathfrak{H}_p$ if, and only if, the function $(1+w)^{-2/p}\varphi(w)$ belongs to H_p, where $\varphi(w)=F(z)$.*

Hille and Tamarkin have proved[11] that the condition $\varphi(w)\in H_p$ is necessary. To define the function $(1+w)^{-2/p}$, we cut the w-plane along the negative real axis from $w=-1$ to $w=-\infty$. When $F(z)$ belongs to $\mathfrak{H}_p$ then its limit function $F(t)$ $[y\to 0,\ x=t]$ belongs to $L_p(-\infty,\ \infty)$, therefore $(1+e^{i\vartheta})^{-2/p}\varphi(e^{i\vartheta})$ to $L_p(-\pi,\ \pi)$. Let $\psi(w)=(1+w)^{-2/p}\varphi(w)$, and $0<q<p/3$. By Hölder's theorem, we have

$$\int_{-\pi}^{\pi}|\psi(re^{i\vartheta})|^q d\vartheta\leq\left(\int_{-\pi}^{\pi}|\varphi(re^{i\vartheta})|^p\right)^{q/p}\left(\int_{-\pi}^{\pi}\frac{d\vartheta}{|1+re^{i\vartheta}|^{2q/(p-q)}}\right)^{1-q/p}.$$

The right side is uniformly bounded for $0<r<1$. Hence $\psi(w)\in H_q$; its limit-function $\psi(e^{i\vartheta})$, however, belongs to $L_p(-\pi,\ \pi)$; hence[12] $\psi(w)\in H_p$. Conversely, let $\psi(w)\in H_p$. From a result due to R. M. Gabriel[13] we deduce that

$$\int_C|\psi(w)|^p|dw|\leqq 2\int_{-\pi}^{\pi}|\psi(e^{i\vartheta})|^p d\vartheta,$$

where C is any circle strictly interior to the unit circle Γ $[|w|=1]$. By Fatou's theorem, this inequality holds when C is a circle touching Γ from within at $w=-1$. Finally, by the transformation $w=(i-z)(i+z)^{-1}$, we deduce that $|F(x+iy)|_p\leqq 2^{2/p}\|\psi(e^{i\vartheta})\|_p$ $[0<y<\infty]$ which proves the lemma. In a similar way we can show that *when $F(z)\in\mathfrak{H}_p$ and $F(t)\in L_q(-\infty,\ \infty)$ $[0<p_q^p\leqq\infty]$ then $F(z)\in\mathfrak{H}_q$.*

LEMMA 3. *Let $0<p\leqq\infty$, let $f_n(z)\in\mathfrak{H}_p$ $[n=1,\ 2,\ \cdots]$, and let $f_n(t)$ be the limit-function of $f_n(z)$. Let $F(t)$ be defined in $(-\infty,\ \infty)$ and $|F(t)-f_n(t)|_p\to 0$ as $n\to\infty$. Then $F(t)$ is equivalent to the limit-function of an element $f(z)$ of $\mathfrak{H}_p$.*

[11] Loc. cit. Lemma 2.5.

[12] V. Smirnoff, C. R. Acad. Sci. Paris vol. 188 (1929) pp. 131–133. A. Zygmund, *Trigonometrical series*, Warsaw, 1935, 7.56(iv).

[13] J. London Math. Soc. vol. 5 (1930) pp. 129–131. Cf. Hille-Tamarkin, Lemmas 2.1 and 2.5.

The proof for $0<p<\infty$ is entirely different from that for $1\leqq p\leqq\infty$, given in a former paper.[14] Let $0<p<1$ and $\rho>0$, and let $\phi(z)\in\mathfrak{H}_p$. Then, for $\rho\leqq y<\infty$, we have $|\phi(z)|\leqq((1/2)\pi\rho)^{-1/p}\ |\phi(t)|_p$.[15] Since $|f_n(t)-f_m(t)|_p\rightarrow 0$ $[m>n\rightarrow\infty]$, taking $\phi(z)=f_m(z)-f_n(z)$ we can deduce that the sequence $\{f_n(z)\}$ converges to an analytic function $f(z)$, uniformly for $-\infty<x<\infty$, $\rho<y<\infty$. Since there is a constant K, independent of n, such that $|f_n(t)|_p\leqq K$, we have $|f_n(x+iy)|_p\leqq K$, and we can deduce that $|f(x+iy)|_p\leqq K$ for any positive y. Hence $f(z)\in\mathfrak{H}_p$. We are left to show that $f(t)$, the limit-function of $f(z)$, is equivalent to $F(t)$ in $(-\infty,\ \infty)$. Given $\epsilon>0$, we have $|(f_m(x)-f_N(x)|_p^p<\epsilon/12$ for $m\geqq N$, fixing N in a suitable way, and $|f_N(x+iy)-f_N(x)|_p^p\leqq\epsilon/6$ for $0<y\leqq\delta=\delta(\epsilon,\ N)$. Hence

$$|f_m(x+iy)-f_m(x)|_p^p\leqq|f_m(x+iy)-f_N(x+iy)|_p^p$$
$$+|f_m(x)-f_N(x)|_p^p+|f_N(x+iy)-f_N(x)|_p^p\leqq\epsilon/3$$

for $m\geqq N$, $0<y\leqq\delta$, since the first term on the right side is not greater than the second term. Given $M>0$, we have

$$\int_{-M}^{M}|f(x)-f_m(x)|^p dx\leqq\int_{-M}^{M}|f(x+iy)-f_m(x+iy)|^p dx$$
$$+|f(x+iy)-f(x)|_p^p+|f_m(x+iy)-f_m(x)|_p^p.$$

The right side is smaller than ϵ for $m\geqq m_0(\epsilon)$, as we see fixing a suitable value for y. Consequently $f(x)=F(x)$ almost everywhere in any finite interval $(-M,\ M)$ and, therefore, in $(-\infty,\ \infty)$. With a slight alteration, the proof holds for $1\leqq p<\infty$.

By the lemma, the necessity of the condition in Theorem 2(a) is evident. For $s_n(t)$ belongs to $L_p(-\infty,\ \infty)$, therefore $s_n(z)$ to $\mathfrak{H}_p$. To prove its sufficiency, we take first $1<p<\infty$. By Theorem 1, there are rational functions $R_n(z)$ such that their only poles lie at $z=\bar{\beta}$ and $z=\beta$ and that $|F(t)-R_n(t)|_p\rightarrow 0$ as $n\rightarrow\infty$. Taking $R_n(z)=s_n(z)+\sigma_n(z)$, where the rational functions s_n and σ_n vanish at infinity and have no poles other than at $z=\bar{\beta}$ or $z=\beta$, respectively, we have $s_n(z)\in\mathfrak{H}_p$, $\overline{\sigma_n(\bar{z})}\in\mathfrak{H}_p$. Denoting by $\mathfrak{H}f$ the Hilbert operator

$$\mathfrak{H}f=\frac{1}{\pi}PV\int_{-\infty}^{\infty}\frac{f(t)dt}{t-x},$$

we have $|\mathfrak{H}f|_p\leqq C_p|f|_p$, $\mathfrak{H}F=iF(x)$ and $\mathfrak{H}s_n=is_n(x)$, $\mathfrak{H}\sigma_n=-i\sigma_n(x)$.[14]

[14] H. Kober, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 421–427.

[15] This can be shown by means of the inequality (73), M. Plancherel and G. Polya, Comment. Math. Helv. vol. 10 (1937–1938) pp. 110–163.

Hence

$$2\,|F(t) - s_n(t)|_p = |iF + \mathfrak{H}F - (iR_n + \mathfrak{H}R_n)|_p$$
$$\leqq |F - R_n|_p + |\mathfrak{H}(F - R_n)|_p \leqq (C_p + 1)\,|F - R_n|_p$$

which tends to zero as $n\to\infty$. Hence $|F(t)-s_n(t)|_p\to 0$ as $n\to\infty$.

Let now $0<p\leqq 1$ and $F(z)\in\mathfrak{H}_p$, let $\beta=-i$, $z=i(1-w)(1+w)^{-1}$ and $\varphi(w)=F(z)$. Given $\epsilon>0$, from the Lemmas 2 and 1 we infer the existence of a polynomial $P(z)$ such that

$$\int_{-\pi}^{\pi} |\varphi(e^{i\vartheta})(1+e^{i\vartheta})^{-2/p} - P(e^{i\vartheta})|^p d\vartheta \leqq \epsilon/4.$$

Hence

$$\int_{-\infty}^{\infty}\left|F(t) - (1+e^{i\vartheta})^{2/p}P\left(\frac{i-t}{i+t}\right)\right|^p dt \leqq \epsilon/2,$$

where $t=\tan\vartheta/2$. Let b be an integer, $p^{-1}<b\leqq 1+p^{-1}$. Then the rational function $\chi(z)=(2i)^b(i+z)^{-b}P\{(i-z)(i+z)^{-1}\}$ has no singularity except at $z=-i$. Since $\chi(t)\in L_p(-\infty,\ \infty)$, we have $|\chi(t)|_p=C<\infty$. Now the function $(1+e^{i\vartheta})^{2/p-b}$ can be approximated by polynomials $Q_m(e^{i\vartheta})$ $[m=1,\ 2,\cdots]$, uniformly for $-\pi\leqq\vartheta\leqq\pi$. Hence, for some m, we have

$$\int_{-\infty}^{\infty}\left|(1+e^{i\vartheta})^{2/p}P\left(\frac{i-t}{i+t}\right) - \chi(t)Q_m\left(\frac{i-t}{i+t}\right)\right|^p dt < \epsilon/2.$$

Thus $|F(t)-\chi(t)Q_m\{(i-t)(i+t)^{-1}\}|_p<\epsilon^{1/p}$. This completes the proof which, slightly altered, holds for $1<p\leqq 2$.

For $p=1,\ 2,\ \infty$, we obtain explicit approximating functions by Theorem 1′ and by the lemma:

Let $1\leqq p\leqq\infty$ and $F(z)\in\mathfrak{H}_p$, let a be an integer and $a\geqq 0$ for $p=1$, $a\geqq 2$ for $p=\infty$, $a\geqq 1$ otherwise; then

$$\int_{-\infty}^{\infty} F(t)\frac{(\alpha-t)^n}{(t-\beta)^{n+a}}\,dt = 0 \quad \textit{for } n = 0,\ 1,\ 2,\cdots.$$

THEOREM 2′. *Let $p=2$, 1, or ∞ and $c=1$, 2, or 0, respectively; let $F(z)\in\mathfrak{H}_p$ and $F(t)$, the limit-function of $F(z)$, be continuous everywhere including infinity when $p=\infty$. Let $s_n(z)$ be defined by*

$$\sum_{k=0}^{n} a_k\frac{(\bar\beta - z)^k}{(z-\beta)^{k+c}}\,[p = 2] \quad \textit{or} \quad \frac{1}{n+1}\sum_{j=0}^{n}\sum_{k=0}^{j} a_k\frac{(\bar\beta - z)^k}{(z-\beta)^{k+c}}\begin{bmatrix} p = 1, \\ p = \infty \end{bmatrix},$$

where

$$a_k = i\frac{\bar{\beta}-\beta}{2\pi}\int_{-\infty}^{\infty} F(t)\frac{(t-\beta)^{k+c-1}}{(\bar{\beta}-t)^{k+1}}\,dt.$$

Then, uniformly for $0 \leqq y < \infty$, $|F(x+iy)-s_n(x+iy)|_p \to 0$ *as* $n\to\infty$.

Applying Theorem 2′ to the components of $g(z)=(1/2)s(1-s)\Gamma((1/2)s)\pi^{-s/2}\zeta(s)$,[16] where $\zeta(s)$ is the Riemann zeta-function and $z=i(1-2s)$, we can deduce the following corollary:

Let $0\leqq a<\infty$, $q=i(1-a)$, $r=i(1+a)$, *let*

$$\vartheta(x) = \sum_{n=-\infty}^{\infty} e^{-n^2\pi x}; \qquad b_0 = \vartheta(1)/2 + (1-a/2)\int_1^{\infty} v^{a/4}\vartheta'(v)\,dv;$$

$$L_n^{(j)}(x) = \sum_{k=0}^{n} C_{n+j,k+j}\frac{(-x)^k}{k!};$$

$$b_n = (-1)^n\int_1^{\infty} v^{a/4}\vartheta'(v)\left\{L_n^{(0)}(\log v)/2 - (a/2)L_n^{(-1)}(\log v)/2\right\}dv, \qquad n>0.$$

Then the series

$$\sum_{n=0}^{\infty} b_n\left\{\left(\frac{q-z}{r+z}\right)^n + \left(\frac{q+z}{r-z}\right)^n\right\}$$

converges to $g(z)$ *uniformly for* $-\infty<x<\infty$, $-a\leqq y\leqq a$, *while it does not converge whenever* $|y|>a$.

The series takes a simple form for $a=0$ (critical line).

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[16] In fact to the function $g_1(z-ia)\in\mathfrak{H}_\infty$, where $g(z)=g_1(z)+g_1(-z)$, $g_1(z)=((1+z^2)/16)\int_1^\infty\{\vartheta(t)-1\}t^{(iz-3)/4}dt-(1/4)-(iz/4)\{\vartheta(1)-1\}$.