TRANSFORMATIONS OF MULTIPLE FOURIER SERIES

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1. **Introduction.** The object of the present paper is the study and characterization of certain classes of factor sequence transformations of multiple Fourier series. A recent moment problem solution[1] by the author and a scheme of summation of multiple Fourier series developed by Bochner[2] are used in the study. The results include and extend known results for single Fourier series.

2. **Definitions and notation.** Let n be a positive integer, fixed but arbitrary. R^n will denote the euclidean n-space. (x), (y), and so on will denote $(x_1, x_2, \cdots, x_n)$, $(y_1, y_2, \cdots, y_n)$, and so on, points of R^n. ν, τ, j, k, s will be used for non-negative integers, and (ν), (τ), and so on will be used for $(\nu_1, \nu_2, \cdots, \nu_n)$, $(\tau_1, \tau_2, \cdots, \tau_n)$, lattice points of R^n. (0) will mean $(0, 0, \cdots, 0)$, and $(x)=(y)$ will mean $x_j=y_j$, $j=1, 2, \cdots, n$. $(k\cdot x)$ will stand for the number $k_1x_1+k_2x_2+\cdots+k_nx_n$, $|x|$ for the number $(x_1^2+x_2^2+\cdots+x_n^2)^{1/2}$. A, I, and λ will be used for functions defined on the lattice points of R^n, and I will be the characteristic function of the lattice points of R^n. E will be the set $E_{(x)}(-\pi\leqq x_j<\pi, j=1, 2, \cdots, n)$. R and t will be used for real numbers. $(x+y)$ will stand for $(x_1+y_1, x_2+y_2, \cdots, x_n+y_n)$, and $B(n)$ for a real constant depending only on n. Φ will be used for a function U^* of bounded variation in the sense of Saks, and if $\Phi(H)=\Phi_1(H)+\Phi_2(H)$ for any Borel set H with $\Phi_1(H)\geqq 0\geqq\Phi_2(H)$ we will write $\int_H f(x)|d\Phi(E)|$ for $\int_H f(x)d\Phi_1(E)-\int_H f(x)d\Phi_2(E)$. When Φ is the Lebesgue measure function we will write $\int_H f(x)dx$ for

$$\int_H f(x)d\Phi(E).$$

We will write $f\in L$ to indicate that $\int_E f(x)dx$ exists, and $f\in C$ to indicate that f is continuous on $\overline{E}$ and $f(x)=f(x+y)$ for all combinations of $y_j=0$ or 2π, $j=1, 2, \cdots, n$. A function f defined over E will be defined over R_n by the extension $f(x)=f(x+y)$ with $y_j=0$ or 2π, $j=1, 2, \cdots, n$.

Let

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[1] L. B. Hedge, *Moment problem for a bounded region*, Bull. Amer. Math. Soc. vol. 47 (1941) pp. 282–285. Referred to later as MP.

[2] S. Bochner, *Summation of multiple Fourier series by spherical means*, Trans. Amer. Math. Soc. vol. 40 (1936) pp. 175–207.

(1) $$\mathfrak{S}(\lambda, A, x) \simeq \sum_{(k)} \lambda(k)A(k)e^{i(k\cdot x)},$$

and

(2) $$\mathfrak{S}_\nu(\lambda, A, x) = \sum_{|k|^2\leqq\nu} \lambda(k)A(k)e^{i(k\cdot x)}.$$

If

(3) $$\lambda(k)A(k) = (2\pi)^{-n}\int_E f(x)e^{-i(k\cdot x)}dx$$

or

(4) $$\lambda(k)A(k) = (2\pi)^{-n}\int_E e^{-i(k\cdot x)}d\Phi(E), \qquad \int_E |d\Phi(E)| < \infty,$$

we will write $\mathfrak{S}(f, x)$ or $\mathfrak{S}(d\Phi, x)$, respectively, for the left side of equation (1), and similarly alter equations (2) and (5).

We write

(5) $$S_R(\lambda, A, x) = \sum_{(k)} \Psi(|k|/R)\lambda(k)A(k)e^{i(k\cdot x)},$$

where
$$\Psi(t) = e^{-t^2},$$

for the Bochner-Gauss[2] spherical means of the sequence (2), and

(6) $$K_R(u) = \sum_{(k)} \Psi\left(\frac{|K|}{R}\right)e^{-i(k\cdot u)} = S_R(I, I, u)$$

for the corresponding kernel.

3. **Spherical summation.** We proceed to some modification (largely notational) of the Bochner summation theory. The transformation which takes (2) into (5) is given by the matrix $T:\|a_{R,\nu}\|$, where $a_{R,\nu}=\Psi(R_\nu/R)-\Psi(R_{\nu+1}/R)$, and $\{R_\nu\}$ is a subsequence of $\{0, 1, 2^{1/2}, 3^{1/2}, 4^{1/2}, \cdots, k^{1/2}, \cdots\}$. We have at once

$$a_{R,\nu} \geqq 0, \qquad \lim_{R\to\infty} a_{R,\nu} = 0, \qquad \sum_{\nu=0}^{\infty} a_{R,\nu} = 1,$$

whence T is a regular Toeplitz transformation.[3]

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[3] See, for example, A. Zygmund, *Trigonometrical series*, Monografje Matematyczne, vol. 5, Warsaw, 1935, pp. 79–86. That R is a continuous variable is unimportant.

The following properties of the summation scheme follow immediately from Bochner's work:

$$(7)\qquad 0 \leqq K_R(u) \leqq M_R < \infty,$$

$$(8)\qquad S_R(f, x) = B(n)\int_E f(u)K_R(u - x)du, \qquad \text{if } f \in L,$$

and,

$$(9)\qquad S_R(f, x) \to f(x)$$

at every point (x) of continuity of f, and uniformly on E if $f \in C$.

4. **Classes of multiple Fourier series.** The theorem of MP may now be given in the following form:

THEOREM. *In order that* $\mathfrak{S}(\lambda, A, x) = \mathfrak{S}(d\Phi, x)$ *with*

$$1.\ \int_E |d\Phi| \leqq M, \qquad or \qquad 2.\ \Phi \geqq 0,$$

it is necessary and sufficient that

$$1.\ \int_E |S_R(\lambda, A, x)|\, dx \leqq M, \qquad or \qquad 2.\ S_R(\lambda, A, x) \geqq 0,$$

and in order that $\mathfrak{S}(\lambda, A, x) = \mathfrak{S}(f, x)$ *with*

$$3.\ f \in L, \qquad or \qquad 4.\ |f| \leqq M, \qquad or \qquad 5.\ f \in C,$$

it is necessary and sufficient that 3. $\{S_R(\lambda, A, x)\}$ *converge in the mean with exponent* 1, *or* 4. $|S_R(\lambda, A, x)| \leqq M$, *or* 5. $\{S_R(\lambda, A, x)\}$ *converge uniformly in* B.

We shall write $\mathfrak{S}(\lambda, A, x) \in S$ to indicate that $\mathfrak{S}(\lambda, A, x) = \mathfrak{S}(d\Phi, x)$ with $\int_E |d\Phi| \leqq M$, and $\mathfrak{S}(\lambda, A, x) \in L$, M, or C, if $\mathfrak{S}(\lambda, A, x) = \mathfrak{S}(f, x)$ with $f \in L$, $f \in M$ or $f \in C$ respectively. We will write $\lambda \in (P, Q)$ to indicate that $\mathfrak{S}(I, A, x) \in P$ implies $\mathfrak{S}(\lambda, A, x) \in Q$.

5. **Transformations.** We begin with the following lemma.

LEMMA 1. $\mathfrak{S}(I, I, x) \in S$.

PROOF. Let $f \in C$. Since $S_R(I, I, x) = K_R(x)$, we have

$$S_R(f, x) = B(n)\int_E f(u)S_R(I, I, u - x)du,$$

but f is continuous, the left-hand member converges uniformly to f, and for any sequence of values of R, $\{|S_R(f, x)|\}$ is bounded uniformly. $\{S_R(f,x)\}$ is a sequence of linear operations whose norms are

$$\int_E |S_R(I, I, x)| \, dx,$$

and by a simple corollary of a theorem of Banach and Steinhaus[4] these must be uniformly bounded, or $\mathfrak{S}(I, I, x) \in S$.

We now have this theorem.

THEOREM 1. *The transformation classes* (S, S), (M, M), (L, L), *and* (C, C) *are identical, and* λ *belongs to each of them if and only if* $\mathfrak{S}(\lambda, I, x) \in S$, *or*

$$\int_E |S_R(\lambda, I, x)| \, dx \leqq M. \tag{10}$$

(A) $\lambda \in (S, S) \rightarrow (10)$. This follows immediately from Lemma 1.

(B) $(10) \rightarrow \lambda \in (S, S)$. From (5) we have

$$\begin{aligned} S_R(\lambda, A, x) &= \sum_{(k)} \Psi\left(\frac{|K|}{R}\right) \lambda(k) A(k) e^{i(k \cdot x)} \\ &= \sum_{(k)} \Psi\left(\frac{|K|}{R}\right) \lambda(k) (2\pi)^{-n} \int_{E(u)} e^{i(k \cdot x - u)} d\Phi(E)^5 \\ &= (2\pi)^{-n} \int_{E(u)} S_R(\lambda, I, x - u) f(u) d\Phi(E), \end{aligned}$$

whence

$$\int_E |S_R(\lambda, A, x)| dx \leqq (2\pi)^{-n} \cdot M \cdot \int_E |d\Phi|.$$

(C) $\lambda \in (M, M) \rightarrow (10)$ and $\lambda \in (C, C) \rightarrow (10)$.

We have immediately in both cases

$$S_R(\lambda, A, x) = B(n) \int_{E(u)} S_R(\lambda, I, x - u) f(u) du$$

and the boundedness of the set $\{S_R(\lambda, A, 0)\}$ implies (10).

(D) $(10) \rightarrow \lambda \in (M, M)$, $(10) \rightarrow \lambda \in (C, C)$, and $(10) \rightarrow \lambda \in (L, L)$.

From (5) and (10) we have

[4] S. Banach, *Théorie des opérations linéaires*, Monografje Matematyczne, vol. 1, Warsaw, 1932, p. 80.

[5] The notation $\int_{E(u)} f(x, u) d\Phi(E)$ is used to indicate the integral over E of f regarded as a function of a point (u).

$$S_R(\lambda, A, x) = \sum_{(k)} \Psi\left(\frac{|K|}{R}\right) A(k)(2\pi)^{-n} \int_{E(u)} e^{-i(k\cdot x - u)} d\Phi(E)$$

$$= (2\pi)^{-n} \int_{E(u)} S_R(I, A, x-u) d\Phi(E),$$

and the boundedness, uniform convergence, or mean convergence of $\{S_R(I, A, x)\}$ implies the same for $\{S_R(\lambda, A, x)\}$.

(E) $\lambda \in (L, L) \rightarrow (10)$.

Suppose (10) does not hold. Then there is a sequence $\{R_m\}$ and a sequence of sets $\{G_m\}$, each G being a finite sum of nonoverlapping cubes of E, such that

$$\int_{G_m} S_{R_m}(\lambda, A, x) dx = (2\pi)^{-n} \int_{E(u)} f(u) \left\{ \int_{G_m} S_{R_m}(\lambda, I, x-u) dx \right\} du$$

is not bounded. But this is a sequence of linear functionals defined over L, of norm

$$\max_u \left| \int_{G_m} S_{R_m}(\lambda, I, x-u) dx \right|,$$

and by the theorem of Banach and Steinhaus[4] there is an $f \in L$ such that

$$\left\{ \int_E S_{R_m}(\lambda, A, x) dx \right\}$$

is unbounded, and hence $\mathfrak{S}(\lambda, A, x) \notin S$, which obviously implies $\mathfrak{S}(\lambda, A, x) \notin L$, contrary to hypothesis.

THEOREM 2. *The transformation classes* (S, L) *and* (M, C) *are identical, and* λ *belongs to each of them if and only if* $\mathfrak{S}(\lambda, I, x) \in L$, *or*

$$\lim_{R, R' \to \infty} \int_E \left| S_R(\lambda, I, x) - S_{R'}(\lambda, I, x) \right| dx = 0. \tag{11}$$

PROOF. (A) $\lambda \in (S, L) \rightarrow (11)$, immediately, from Lemma 1.

(B) $(11) \rightarrow \lambda \in (S, L)$.

Since $S_R(\lambda, A, x) = (2\pi)^{-n} \int_{E(u)} S_R(\lambda, I, x-u) d\Phi(E)$ it follows that (11) implies the convergence in the mean of $\{S_R(\lambda, A, x)\}$.

(C) $(11) \rightarrow \lambda \in (M, C)$.

Since $S_R(\lambda, A, x) = (2\pi)^{-n} \int_{E(u)} S_R(\lambda, I, x-u) f(u) du$, with $|f| \leqq M$ by hypotheses, the uniform convergence of the sequence $\{S_R(\lambda, A, x)\}$ follows immediately from (11).

(D) $\lambda \in (M, C) \rightarrow (11)$.

Writing

$$S_R(\lambda, A, x) = (2\pi)^{-n} \int_{E(u)} S_R(\lambda, I, x - u) f(u) du,$$

we have

$$S_R(\lambda, A, 0) = (2\pi)^{-n} \int_E S_R(\lambda, I, u) f(u) du = U_f(S_R(\lambda, I, u))$$

which defines a linear function U_f. The sequence of values $\{ U_f(S_R(\lambda, I, u)) \}$ converges for every bounded f, that is, $\{ S_R(\lambda, I, u) \}$ converges weakly. But L is weakly complete,[6] whence there is an $F \in L$ such that

$$\lim_{R\to\infty} \int_E \{ S_R(\lambda, I, u) - F(u) \} f(u) du = 0$$

for every bounded f. Consider now the two series

$$\mathfrak{S}(F, u) \simeq \sum_{(k)} C(k) e^{i(k \cdot u)}$$

and

$$\mathfrak{S}(\lambda, I, u) \simeq \sum_{(k)} \lambda(k) e^{i(k \cdot u)},$$

and note that

$$S_R(\lambda, I, u) = \sum_{(k)} \Psi\left(\frac{|K|}{R}\right) \lambda(k) e^{i(k \cdot u)}.$$

Let

$$U_N(g) = \int_E g(u) e^{-i(N \cdot u)} du,$$

which is linear on L. Now

$$\lim_{R\to\infty} U_N(S_R(\lambda, I, u)) = \lim_{R\to\infty} \Psi\left(\frac{|N|}{R}\right) \lambda(N) = \lambda(N)$$
$$= U_N(F) = C(N)$$

whence

$$S_R(\lambda, I, u) = S_R(F, u).$$

But

[6] S. Banach, *Théorie des opérations linéaires*, Monografje Matematyczne, vol. 1, Warsaw, 1932, p. 141.

$$\lim_{R\to\infty} \int_E | S_R(F, u) - F(u) | du = 0$$

which, in view of the preceding equality, implies (11).

THEOREM 3. *The transformation classes (S, M), (L, C), and (L, M) are identical, and λ belongs to each of them if and only if $\mathfrak{S}(\lambda, I, x) \in M$, or*

$$|S_R(\lambda, I, x)| \leqq M. \tag{12}$$

PROOF. (A) $\lambda \in (S, M) \rightarrow (12)$ follows immediately from Lemma 1.

(B) $(12) \rightarrow \lambda \in (S, M)$.

Since

$$S_R(\lambda, A, x) = \int_{E(u)} S_R(\lambda, I, x - u) d\Phi(E),$$

we have from (12)

$$| S_R(\lambda, A, x) | \leqq M \int_E | d\Phi(E) |, \quad \text{and} \quad \mathfrak{S}(\lambda, A, x) \in M.$$

(C) $(12) \rightarrow \lambda \in (L, C)$.

We have immediately

$$| S_R(\lambda, A, x) - S_{R'}(\lambda, A, x) | \leqq M \int_E | S_R(I, A, x) - S_{R'}(I, A, x) | dx$$

where $\mathfrak{S}(\lambda, I, x) = \mathfrak{S}(f, x)$, and $|f| \leqq M$. The integral on the right approaches 0 as R and R' approach infinity, whence $\mathfrak{S}(\lambda, A, x) \in C$.

(D) $\lambda \in (L, C) \rightarrow (12)$.

We write immediately

$$(2\pi)^n S_R(\lambda, A, x) = \int_{E(u)} S_R(\lambda, I, x - u) f(u) du$$

and (12) follows from a theorem of Steinhaus and Banach.[4]

(E) $\lambda \in (L, C) \rightarrow \lambda \in (L, M)$ is obvious.

(F) $\lambda \in (L, M) \rightarrow \lambda \in (L, C)$.

Since

$$S_R(\lambda, A, x) = (2\pi)^{-n} \int_{E(u)} S_R(\lambda, I, x - u) f(u) du$$

and $|S_R(\lambda, A, x)| \leqq M(f)$ by hypothesis, it follows that $S_R(\lambda, A, x)$ exists for every $f \in L$, and $\{S_R(\lambda, A, 0)\}$ is a sequence of linear functionals on L, whose norms are

$$\operatorname*{ess\,sup}_{u} \left| S_R(\lambda, I, u) \right| = \max_{u} \left| S_R(\lambda, I, u) \right|,$$

and for every $f \in L$

$$\left| S_R(\lambda, A, 0) \right| \leqq M(f),$$

whence, by the theorem of Steinhaus and Banach[4] the norms are uniformly bounded. Hence $\lambda \in (L, M) \rightarrow (12) \rightarrow \lambda \in (L, C)$.

THEOREM 4. *λ belongs to the transformation class (S, C) if and only if $\mathfrak{S}(\lambda, I, x) \in C$, or*

$$(13) \qquad \lim_{R, R' \to \infty} \left| S_R(\lambda, I, x) - S_{R'}(\lambda, I, x) \right| = 0, \textit{ uniformly in } x \in E.$$

PROOF. (A) $\lambda \in (S, C) \rightarrow (13)$ immediately from Lemma 1.

(B) $(13) \rightarrow \lambda \in (S, C)$.

Since

$$S_R(\lambda, A, x) - S_{R'}(\lambda, A, x) = (2\pi)^{-n} \int_{E(u)} S_R(\lambda, I, x - u) d\Phi(E)$$
$$- (2\pi)^{-n} \int_{E(u)} S_{R'}(\lambda, I, x - u) d\Phi(E),$$

we may write

$$\left| S_R(\lambda, A, x) - S_{R'}(\lambda, A, x) \right|$$
$$\leqq \sup_{x \in E} \left| S_R(\lambda, I, x) - S_{R'}(\lambda, I, x) \right| \int_E \left| d\Phi(E) \right|$$

and the uniform convergence of $\{S_R(\lambda, A, x)\}$ follows from that of $\{S_R(\lambda, I, x)\}$.

6. **Conclusion.** Of the factor sequence transformations among the classes S, L, M, and C, all of those characterizable in terms of these classes applied to $\mathfrak{S}(\lambda, I, x)$ have been discussed. The class (L, M) of transformations does not exist in a proper sense since, by Theorem 3, its range is a subset of C contained in M. The results of the paper may be taulated as follows:

$$\mathfrak{S}(\lambda, I, x) \in S \equiv \lambda \in (S, S) \equiv (L, L) \equiv (M, M) \equiv (C, C),$$
$$\mathfrak{S}(\lambda, I, x) \in L \equiv \lambda \in (S, L) \equiv (M, C),$$
$$\mathfrak{S}(\lambda, I, x) \in M \equiv \lambda \in (S, M) \equiv (L, C),$$
$$\mathfrak{S}(\lambda, I, x) \in C \equiv \lambda \in (S, C).$$

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