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California Institute of Technology

ON THE QUADRIC OF LIE

SU-CHENG CHANG

The most important quadric attached to an ordinary point of a non-ruled and nondegenerate surface is, perhaps, the quadric of Lie. The characteristic curve of the quadric of Lie varying along an asymptotic curve of the surface decomposes into an asymptotic tangent and two edges of the quadrilateral of Demoulin.[1] In this note we propose to determine whether the characteristic curve of the quadric of Lie may decompose into two conics when the quadric of Lie varies along certain curves of the surface. The answer is positive.

Let (u, v) be the asymptotic net of a surface (M) and (M, M_1, M_2, M_3) its normal tetrahedron of Cartan, MM_1, MM_2 being the two asymptotic tangents and MM_3, M_1M_2 being the directrices of Wilczynski. Except for a projective transformation the surface (M) is determined by the system

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[1] A. Demoulin, *Sur la théorie des lignes asymptotiques*, C. R. Acad. Sci. Paris vol. 147 (1908) pp. 413–415, *Sur la quadrique de Lie*, ibid. pp. 493–496, *Sur quelques propriétés des surfaces courbes*, ibid. pp. 565–568.

$$
(1)\quad
\begin{aligned}
\frac{\partial M}{\partial u} &= (1/2)\frac{\partial \log \gamma}{\partial u} M + M_1, \\
\frac{\partial M_1}{\partial u} &= B^2 M - (1/2)\frac{\partial \log \gamma}{\partial u} M_1 + \beta M_2, \\
\frac{\partial M_2}{\partial u} &= KM + (1/2)\frac{\partial \log \gamma}{\partial u} M_2 + M_3, \\
\frac{\partial M_3}{\partial u} &= A^2\beta M + KM_1 + B^2 M_2 - (1/2)\frac{\partial \log \gamma}{\partial u} M_3, \\
\frac{\partial M}{\partial v} &= (1/2)\frac{\partial \log \beta}{\partial v} M + M_2, \\
\frac{\partial M_1}{\partial v} &= \overline{K} M + (1/2)\frac{\partial \log \beta}{\partial v} M_1 + M_3, \\
\frac{\partial M_2}{\partial v} &= A^2 M - (1/2)\frac{\partial \log \beta}{\partial v} M_2 + \gamma M_1, \\
\frac{\partial M_3}{\partial v} &= B^2\gamma M + A^2 M_1 + \overline{K} M_2 - (1/2)\frac{\partial \log \beta}{\partial v} M_3,
\end{aligned}
$$

where M denotes a point of the surface with the coordinates M^i $(i=1, 2, 3, 4)$ and

$$
2K = \beta\gamma - \frac{\partial^2 \log \beta}{\partial u \partial v}, \qquad 2\overline{K} = \beta\gamma - \frac{\partial^2 \log \gamma}{\partial u \partial v} \cdot
$$

The conditions of integrability of the system (1) are found to be

$$
(2)\quad
\begin{aligned}
&\frac{\partial A^2}{\partial u} = K\frac{\partial(\log \beta K)}{\partial v}, \qquad \frac{\partial B^2}{\partial v} = \overline{K}\frac{\partial(\log \gamma \overline{K})}{\partial u}, \\
&\qquad A\frac{\partial(A\beta)}{\partial v} = B\frac{\partial(B\gamma)}{\partial u} \cdot
\end{aligned}
$$

As the coordinates of any point in space can be expressed in the form

$$
(3)\qquad P = y_1 M + y_2 M_1 + y_3 M_2 + y_4 M_3,
$$

(y_1, y_2, y_3, y_4) being the local coordinates of P with respect to the tetrahedron $\{M, M_1, M_2, M_3\}$, we have the equation of the quadric of Lie at M, namely,

$$
(4)\qquad y_1 y_4 - y_2 y_3 = 0.
$$

Suppose that the point P be fixed in space, from (1) and (3) it follows that

$$
\begin{aligned}
\frac{\partial y_1}{\partial u} &= -(1/2)\frac{\partial \log \gamma}{\partial u} y_1 - B^2 y_2 - K y_3 - A^2\beta y_4, \\
\frac{\partial y_2}{\partial u} &= -y_1 + (1/2)\frac{\partial \log \gamma}{\partial u} y_2 - K y_4, \\
\frac{\partial y_3}{\partial u} &= -\beta y_2 - (1/2)\frac{\partial \log \gamma}{\partial u} y_3 - B^2 y_4, \\
\frac{\partial y_4}{\partial u} &= -y_3 + (1/2)\frac{\partial \log \gamma}{\partial u} y_4, \\
\frac{\partial y_1}{\partial v} &= -(1/2)\frac{\partial \log \beta}{\partial v} y_1 - \overline{K} y_2 - A^2 y_3 - B^2\gamma y_4, \\
\frac{\partial y_2}{\partial v} &= -(1/2)\frac{\partial \log \beta}{\partial v} y_2 - \gamma y_3 - A^2 y_4, \\
\frac{\partial y_3}{\partial v} &= -y_1 + (1/2)\frac{\partial \log \beta}{\partial v} y_3 - \overline{K} y_4, \\
\frac{\partial y_4}{\partial v} &= -y_2 + (1/2)\frac{\partial \log \beta}{\partial v} y_4.
\end{aligned} \tag{5}
$$

Differentiating (4) along a curve $v=v(u)$ of the surface (M) and making use of (5), we obtain that the characteristic of the quadric of Lie along this curve is given by (4) and

$$\beta(y_2^2 - A^2 y_4^2) + \frac{dv}{du}\gamma(y_3^2 - B^2 y_4^2) = 0. \tag{6}$$

The latter represents two planes when and only when

$$\frac{dv}{du} = -\frac{\beta A^2}{\gamma B^2}. \tag{7}$$

Therefore *there are curves L of a one-parameter system on a surface along each of which two consecutive quadrics of Lie intersect in two conics C_1 and C_2.* It is easily seen that the planes of these conics at M are

$$B y_2 - A y_3 = 0 \tag{8$_1$}$$

and

$$(8_2) \qquad By_2 + Ay_3 = 0,$$

respectively.

In order that *the curves L should be indeterminate it is necessary and sufficient that the surface in consideration is of Demoulin and Godeaux.*[2]

In general *the conics C_1 and C_2 have M and M_3 in common.*

Noticing that the vertices of the quadrilateral of Demoulin are

$$(9) \qquad D_{\epsilon\epsilon'} = \epsilon\epsilon' ABM + \epsilon AM_1 + \epsilon' BM_2 + M_3,$$

where $\epsilon = \pm 1$, $\epsilon' = \pm 1$ and that the planes (8_1) and (8_2) pass through the two diagonals of the quadrilateral of Demoulin, respectively, we infer that *the two conics C_1 and C_2 can be constructed by means of the quadric of Lie and the quadrilateral of Demoulin.*

The second quadric Φ_1, namely, the associate quadric,[3] in the sequence of Godeaux[4] originally defined in virtue of the representation in S_5, is given by the equation

$$(10) \qquad y_1^2 - B^2 y_2^2 - A^2 y_3^2 + A^2 B^2 y_4^2 + \frac{N}{\beta\gamma}(y_1 y_4 - y_2 y_3) = 0$$

where

$$N = B(B\gamma)_u = A(A\beta)_v.$$

It is obvious that *each of the planes (8_i) $(i = 1, 2)$ intersects the associate quadric in a conic which is tangent to the conic C_i $(i = 1, 2)$ at two opposite vertices of the quadrilateral of Demoulin.*

When the point M varies along a curve L of the system the conics C_1, C_2 and their consecutive conics always lie on the quadric of Lie at M.

In fact, a point on the conic C_1 at M can be given parametrically in the form $(AB\rho^2, A\rho, B\rho, 1)$, where ρ is a parameter. By means of (5) we can expand the local coordinates of the points on the consecu-

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[2] L. Godeaux, *Sur une classe de surfaces*, Bulletin de l'Académie royale de Belgique vol. 18 (1932) pp. 1015–1025.

[3] B. Su, *On the surfaces whose asymptotic curves belong to linear complexes*, I, Tôhoku Math. J. vol. 40 (1935) pp. 408–420; II, ibid. pp. 433–448; III, ibid. vol. 41 (1935) pp. 1–9; IV, ibid. pp. 203–215; V, Science Reports of the Tôhoku Imperial University vol. 25 (1935) pp. 601–633; VI, ibid. pp. 634–642; *On the surface whose Lie quadrics all touch a fixed plane*, Science Reports of the National University of Chekiang vol. 2 (1936) pp. 39–51. See also L. Godeaux, *Remarques sur les quadriques associées aux points d'une surface*, Journal of the Chinese Mathematical Society vol. 2 (1937) pp. 1–5.

[4] L. Godeaux, *La théorie des surfaces et l'espace réglé*, Paris, 1934. A new definition of the sequence of Godeaux is given by the present author. See S. C. Chang, *Some theorems on ruled surfaces*, appear in Science Record, Academia Sinica vol. 1 (1942).

tive conic of C_1 into the following series:

$$
(11)\quad
\begin{aligned}
y_1 &= AB\rho^2 + \Bigg[-(1/2)\left(\frac{\partial \log \gamma}{\partial u} - \frac{\beta A^2}{\gamma B^2}\frac{\partial \log \beta}{\partial v}\right) AB\rho^2 - AB^2\rho \\
&\qquad - KB\rho - \frac{\beta A^2}{\gamma B^2}(\overline{K}A\rho + A^2B\rho)\Bigg]\Delta u + (\Delta u^2),\\
y_2 &= A\rho + \Bigg[-AB\rho^2 + (1/2)\frac{\partial \log \gamma}{\partial u}A\rho - K \\
&\qquad - \frac{\beta A^2}{\gamma B^2}\left(-(1/2)\frac{\partial \log \beta}{\partial v}A\rho - B\gamma\rho - A^2\right)\Bigg]\Delta u + (\Delta u^2),\\
y_3 &= B\rho + \Bigg[-\beta A\rho - (1/2)\frac{\partial \log \gamma}{\partial u}B\rho - B^2 \\
&\qquad - \frac{\beta A^2}{\gamma B^2}\left(-AB\rho^2 + (1/2)\frac{\partial \log \beta}{\partial v}B\rho - \overline{K}\right)\Bigg]\Delta u + (\Delta u^2),\\
y_4 &= 1 + \Bigg[-B\rho + (1/2)\frac{\partial \log \gamma}{\partial u} \\
&\qquad - \frac{\beta A^2}{\gamma B^2}\left(-A\rho + (1/2)\frac{\partial \log \beta}{\partial v}\right)\Bigg]\Delta u + (\Delta u^2).
\end{aligned}
$$

Substituting (11) into (4) gives that the coefficients of the terms $(\Delta u)^0$ and $(\Delta u)^1$ vanish for any value of ρ. A similar result holds for the conic C_2, which completes the proof.

NATIONAL UNIVERSITY OF CHEKIANG