In fact if we let

$$T(n) = \sum_{\delta | n} F_{2\delta}$$

then

$$T(n) - 4T(n-1) + 11T(n-3) - 29T(n-6) + \cdots$$
$$= \begin{cases} (-1)^k k F_{2k-1} - F_{2k} & \text{if } n = k(k-1)/2 \\ 0 & \text{otherwise.} \end{cases}$$

Here the mth term of the sequence

$$1, 4, 11, 29, 76, 199, \cdots$$

is

$$F_{2m} + F_{2m-2}.$$

UNIVERSITY OF CALIFORNIA

ON PARTICULAR SOLUTIONS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS

K. L. NIELSEN AND B. P. RAMSAY

1. **Introduction.** The boundary value and characteristic value problems are classical questions in the theory of partial differential equations of elliptic type. A method for actual solution of these problems consisting of approximations by expressions $W_n = \sum_{\nu=1}^{n} \alpha_\nu^{(n)} \phi_\nu(x, y)$, where $\phi_\nu(x, y)$ are *particular solutions of the considered differential equation*, has been given by Bergman (see [1]).[1] Here the $\alpha_\nu^{(n)}$ are constants which are to be determined by the requirement that the values of W_n on the boundary approximate the given data (for details see [1]).[2]

In applying this method it is important for practical purposes to obtain a simple procedure for the construction of the particular solu-

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[1] The numbers in the brackets refer to the bibliography.

[2] This method is in a certain sense the reverse of the Rayleigh-Ritz method in which the approximating expressions satisfy the boundary conditions but do not satisfy the given equation.

tions. In this connection Bergman [2] has proved that to every equation

$$L(U) = U_{z\bar{z}} + aU_z + bU_{\bar{z}} + cU = 0, \tag{1}$$

where a, b, c are functions of $z = x+iy$ and $\bar{z} = x-iy$ and $U_z = \partial U/\partial z$, $U_{\bar{z}} = \partial U/\partial \bar{z}$, $U_{z\bar{z}} = \partial^2 U/\partial z \partial \bar{z}$, there exist functions $E(z, \bar{z}, t)$ such that

$$P(f) = \int_{-1}^{+1} E(z, \bar{z}, t) f(z(1 - t^2)/2) dt/(1 - t^2)^{1/2}, \tag{2}$$

where f is an arbitrary analytic function of one complex variable, will be a particular solution of $L(U) = 0$.[3]

To expedite the numerical computation for practical problems we further desire that E has a simple structure so that we may easily evaluate the integral (2) for arbitrary values of z, $\bar{z}$. In this note we shall determine certain types of equations, $L(U) = 0$, for which E has the simple form

$$E(z, \bar{z}, t) = \exp(Nt^n + Mt^m), \tag{3}$$

where $N = N(z, \bar{z})$ and $M = M(z, \bar{z})$. We shall further show how M and N may be determined from the coefficients a, b, c.

We note further that our results can also be successfully used for the study of the singularities of the functions U satisfying (1). In our case it is possible to show that the functions U possess certain singularities which may be characterized independently of the representation (2) by the property that these U satisfy certain ordinary differential equations in addition to $L(U) = 0$.

[3] The equation (1) is equivalent to the system of two equations $\Delta U^{(1)}/4 + AU_x^{(1)}/2 + BU_y^{(1)}/2 + CU_x^{(2)}/2 + DU_y^{(2)}/2 + c_1U^{(1)} - c_2U^{(2)} = 0$, $\Delta U^{(2)}/4 - CU_x^{(1)}/2 - DU_y^{(1)}/2 + AU_x^{(2)}/2 + BU_y^{(2)}/2 + c_2U^{(1)} + c_1U^{(2)} = 0$ where $U = U^{(1)} + iU^{(2)}$, $c = c_1 + ic_2$, $A = [(a+\bar{a}) + (b+\bar{b})]/2$, $B = [(a-\bar{a}) - (b-\bar{b})]/2i$, $C = -[(a-\bar{a}) + (b-\bar{b})]/2i$, $D = [(a+\bar{a}) - (b+\bar{b})]/2$. If $a = \bar{b}$ and c is real then these equations are completely independent of each other and it is only necessary to take the real part of (2) (see [**1**, p. 540]).

If $a = b = c = 0$; that is, if $L(U) = \Delta U = 0$, then we may take $E = 1$. That is to say that every analytic function of one complex variable is a complex harmonic function. Thus analogous to the case of complex harmonic functions the totality of solutions of the equation $L(U) = 0$ is the sum of two subclasses obtained by the operator P; one from analytic functions and the other from anti-analytic functions of one complex variable. Using these results and the theorem of Runge, Bergman has further shown that the representation (2) yields a method for constructing a denumerable set, S, of particular solutions (which is independent of the domain) which is complete for the totality of solutions in every star-domain with the center at the origin (see [**1**]).

2. **The equation $L(U)=0$ for which (2) with E of the form (3) is a class of particular solutions.** As is easily seen the equation (1) can always be transformed into the equation

$$L'(V) = V_{z\bar{z}} + BV_{\bar{z}} + CV = 0 \tag{4}$$

(see also [2, p. 1172]). Thus it suffices to consider this equation. To simplify the notation in the following theorem we shall use $p^{(\nu)}$ to denote an arbitrary function of z *only* and $q^{(\nu)}$ to denote an arbitrary function of $\bar{z}$ *only*; we shall thus omit the arguments. k_ν shall denote constants and $p_z^{(\nu)}=dp^{(\nu)}/dz$, $q_{\bar{z}}^{(\nu)}=dq^{(\nu)}/d\bar{z}$. f shall denote an arbitrary analytic function of one complex variable.

THEOREM I. *In the equation $L'(V)=0$ let the coefficients B and C have the following forms*

$$\begin{array}{llll}
\mathrm{I}_1: & B = 0 & \textit{and} & C = -\,q^{(1)}q_{\bar{z}}^{(1)}/2,\\
\mathrm{I}_2: & B = -\,p_z^{(1)} & \textit{and} & C = -\,q^{(2)}q_{\bar{z}}^{(2)}/2,\\
\mathrm{II}: & B = 0 & \textit{and} & C = -\,3k_3zq_{\bar{z}}^{(3)}/4 - q^{(3)}q_{\bar{z}}^{(3)}/2,\\
\mathrm{III}: & B = -\,k_1 & \textit{and} & C = -\,q_{\bar{z}}^{(4)}q^{(4)}/2,\\
\mathrm{IV}_1: & B = -\,q^{(5)} & \textit{and} & C = (dB/d\bar{z})/2,\\
\mathrm{IV}_2: & B = -\,q^{(6)} - p_z^{(2)} & \textit{and} & C = -\,q_{\bar{z}}^{(6)}/2,\\
\mathrm{V}: & B = 2k_2z + q^{(7)} & \textit{and} & C = q_{\bar{z}}^{(7)}/2.
\end{array} \tag{5}$$

Then every function

$$V = \int_{-1}^{+1} \exp\,(Nt^n + Mt^m)f(z(1-t^2)/2)dt/(1-t^2)^{1/2} \tag{6}$$

is a particular solution of (4) *and moreover, in the respective cases*

(7)

	N	M	n	m
I_1	$z^{1/2}q^{(1)}$	0	1	0
I_2	$z^{1/2}q^{(2)}$	$p^{(1)}$	1	0
II	$k_3z^{3/2}$	$-\,3k_3z^{3/2}/2 - z^{1/2}q^{(3)}$	3	1
III	k_1z	$z^{1/2}q^{(4)}$	2	1
IV_1	$zq^{(5)}$	0	2	0
IV_2	$zq^{(6)}$	$p^{(2)}$	2	0
V	k_2z^2	$-\,2k_2z^2 - zq^{(7)}$	4	2

On the other hand if (6) *with n, m, and M given by* (7) *are solutions of $L'(V)=0$ for an arbitrary f then the coefficients B and C can be represented by the forms* (5).[4]

PROOF. The proof of this theorem is obtained by considering special cases of the general form of E, (3). A given form of E will determine a solution, V, of the partial differential equation, $L'(V)=0$, provided the coefficients B and C, satisfy the equation

$$G(E) = (1-t^2)(E_{t\bar z}) - E_{\bar z}/t + 2zt(E_{z\bar z} + BE_{\bar z} + CE) = 0 \tag{8}$$

(see [2, p. 1171]).

With the given form (3) we then desire to determine N, M, B and C in such a way that this equation is satisfied. The required derivatives are

$$\left\{\begin{aligned} E_{\bar z} &= e^{Nt^n+Mt^m}\left[t^nN_{\bar z} + t^mM_{\bar z}\right],\\ E_t &= e^{Nt^n+Mt^m}\left[t^{m-1}mM + t^{n-1}nN\right],\\ E_z &= e^{Nt^n+Mt^m}\left[t^mM_z + t^nN_z\right],\\ E_{t\bar z} &= e^{Nt^n+Mt^m}\left[t^{m-1}mM_{\bar z} + t^{2m-1}mMM_{\bar z} + t^{n-1}nN_{\bar z}\right.\\ &\qquad \left. + t^{n+m-1}(mMN_{\bar z} + nNM_{\bar z}) + t^{2n-1}nNN_{\bar z}\right],\\ E_{z\bar z} &= e^{Nt^n+Mt^m}\left[t^mM_{z\bar z} + t^{2m}M_zM_{\bar z} + t^nN_{z\bar z}\right.\\ &\qquad \left. + t^{n+m}(M_zN_{\bar z} + M_{\bar z}N_z) + t^{2n}N_zN_{\bar z}\right].\end{aligned}\right. \tag{9}$$

These values are to be substituted into (8). When the substitution is made and the equation is divided by the common factor $\exp(Nt^n+Mt^m)$ we have an equation which is zero for an arbitrary t; thus the coefficients of each power of t must vanish. This gives us the following system of equations

$$\left\{\begin{array}{ll} t^{m-1}: & (m-1)M_{\bar z} = 0\\ t^{2m-1}: & mMM_{\bar z} = 0,\\ t^{n-1}: & (n-1)N_{\bar z} = 0,\\ t^{n+m-1}: & mMN_{\bar z} + nNM_{\bar z} = 0,\\ t^{2n-1}: & nNN_{\bar z} = 0,\\ t^{m+1}: & (2Bz-m)M_{\bar z} + 2zM_{z\bar z} = 0,\\ t^{2m+1}: & 2zM_zM_{\bar z} - mMM_{\bar z} = 0,\\ t^{n+1}: & (2Bz-n)N_{\bar z} + 2zN_{z\bar z} = 0,\\ t^{n+m+1}: & -mMN_{\bar z} - nNM_{\bar z} + 2zM_{\bar z}N_z + 2zM_{\bar z}N_z = 0,\\ t^{2n+1}: & -nNN_{\bar z} + 2zN_{\bar z}N_z = 0,\\ t: & 2zC = 0.\end{array}\right. \tag{10}$$

[4] This theorem includes the special cases discussed by Bergman in [2].

For various values of m and n this system will take different forms. For many values trivial[5] solutions arise but for others non-trivial solutions exist. We thus obtain the various cases of the theorem.

We shall develop the case for $m=2$ and $n=4$. The system (10) then is

$$(11)\quad \begin{cases} t^9: & N_{\bar z}(2zN_z - 4N) = 0, \\ t^7: & M_{\bar z}(2zN_z - 4N) + 4NN_{\bar z} - 2MN_{\bar z} + 2zN_{\bar z}M_z = 0, \\ t^5: & 2MN_{\bar z} + 4NM_{\bar z} - 2MM_{\bar z} - 4N_{\bar z} + 2zM_zM_{\bar z} \\ & \qquad\qquad + 2zN_{z\bar z} + 2BzN_{\bar z} = 0, \\ t^3: & 2MM_{\bar z} + 3N_{\bar z} - 2M_{\bar z} + 2zM_{z\bar z} + 2BzM_z = 0, \\ t: & M_{\bar z} + 2Cz = 0. \end{cases}$$

From the first equation of (11) it appears that we must consider two cases: $N_{\bar z}=0$ and $N_{\bar z}\neq 0$. Let us consider the first case; then this equation is satisfied. The second equation becomes

$$(12)\qquad M_{\bar z}(2zN_z - 4N) = 0.$$

We need not consider $M_{\bar z}=0$ as with $N_{\bar z}=0$ this would reduce to a trivial case; we thus take $M_{\bar z}\neq 0$ and from (12) we get

$$2zN_z - 4N = 0.$$

Solving this equation we obtain

$$(13)\qquad N = kz^2.$$

The third equation of (11) then reads

$$4NM_{\bar z} - 2MM_{\bar z} + 2zM_zM_{\bar z} = 0$$

and since $M_{\bar z}\neq 0$ and by (13) it becomes

$$2kz^2 - M + zM_z = 0.$$

The solution of this equation is

$$(14)\qquad M = -(2kz^2 + zq).$$

We now consider the fourth equation of (11) which reads

$$(15)\qquad 2MM_{\bar z} - 2M_{\bar z} + 2zM_{z\bar z} + 2BzM_{\bar z} = 0.$$

From (14) we get $M_{\bar z} = -zq_{\bar z}$ and $M_{z\bar z} = -q_{\bar z}$; substituting these into the equation (15) we can solve for B, getting

$$(16)\qquad B = 2kz + q.$$

[5] By a trivial solution we mean a solution which gives $B=C=M=N=0$.

The last equation of (11) reads

$$M_z + 2C_z = 0$$

so that we get

$$C = q_z/2. \tag{17}$$

We thus have case V of the theorem. The other cases can be proved in a similar manner.[6] It is easily seen that all the cases satisfy equation (8).

It remains to prove that if (2) is a solution of (4); that is, $L(P(f))=0$, for every f then $G(E)=0$. Let $f=(z(1-t^2)/2)^\mu$, then by taking the derivatives of $V=P(f)$ and substituting into (4) we see that it reduces to prove that from

$$\int_{-1}^{+1} \frac{(1-t^2)^\mu G(E)}{(1-t^2)^{1/2}}\, dt = 0, \qquad \mu = 0, 1, 2, \cdots, \tag{18}$$

it follows that $G(E)=0$. This follows directly from the fact that in the integral $G(E)$ is an even function of t and the completeness of the system t^μ.

3. **A discussion of the form of** E, (3). The question of whether or not Theorem I gives all the partial differential equations (4) which have solutions (2) with E in the form (3) has not been completely answered. However, the number of non-trivial solutions for B and C is definitely limited and it is possible to give certain relations between m and n in this case by studying the relations between the exponents of t in E. For certain values of m and n, $C=0$ and for others only trivial solutions can result.

We proceed to analyze the system (10) by comparing the exponents of t. The last equation of this system shows that in general C is zero unless m and n have values which will make at least one of the other exponents equal to 1. In order to avoid duplication of possible cases we may, without any loss of generality, make the following assumptions on the values of m and n:

(19) we shall take $n<m$ if n and m have different signs and $|n|>|m|$ in all other cases.

We thus formulate the conditions for m and n for which $C=0$ and in Theorem III, conditions for which B and C do not vanish simultaneously.

THEOREM II. (a) *When* $n>0$ *and* $m>0$, $C=0$ *unless* $m=1$ *or* $m=2$.

[6] A typewritten copy of the enlarged proof with a discussion of all the cases is available at the Brown University Library.

(b) *When* $n<0$ *and* $m<0$, $C=0$. (c) *When* $n<0$ *and* $m>0$, $C=0$ *unless* $m=1$, $m=2$, *or* $n=-m$.

The proof of this theorem follows directly by comparing the exponents of t in (10).

THEOREM III. *The non-trivial solutions for B and C are possible*
(i) *for all m only if* $n=2$, $n=m+2$, $n=m-2$, $n=-m$, $n=2-m$, $n=m+1$, $n=1$, $n=m/2$, $n=(m+2)/2$;
(ii) *for all n only if* $m=0$ *or* $m=2$.

PROOF. Consider the coefficient of t^{2n-1}. If no other exponent is equal to $2n-1$ then we have $nNN_{\bar{z}}=0$. By (19) we have $n\neq 0$ and thus for this equation to be satisfied $N_{\bar{z}}=0$. Consider the coefficient of t^{n+m-1}. If no other exponent is equal to $n+m-1$ then using the fact that $N_{\bar{z}}=0$ this coefficient is $nNM_{\bar{z}}=0$ and since $n\neq 0$ we must have $M_{\bar{z}}=0$ for this equation to be satisfied. But with $N_{\bar{z}}=0$ and $M_{\bar{z}}=0$ we see by substitution into (10) that $B=C=0$. Hence the non-trivial solutions for B and C for general values of m and n are possible only if either of the exponents $n+m-1$ or $2n-1$ is equal to some other exponent. Checking all these possibilities and using (19) we obtain the conditions of the theorem.

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UNIVERSITY OF KENTUCKY