A NOTE ON COMPLEMENTARY SUBSPACES IN A RIEMANNIAN SPACE

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1. **Introduction.**[2] Let

$$(1.1)\qquad x^\kappa = x^\kappa(u^a), \quad \kappa, \lambda, \mu, \nu = 1, \cdots, n; a, b, \cdots, f = 1, \cdots, m,$$

be the equations of a V_m in a V_n with fundamental tensor $g_{\lambda\kappa}$ and let

$$(1.2)\qquad B_a^{\kappa} = \partial_a x^{\kappa} \equiv \frac{\partial x^\kappa}{\partial u^a}\cdot$$

Then the fundamental tensor and curvature tensor of V_m in V_n are, respectively,

$$(1.3)\qquad 'g_{cb} = g_{\lambda\kappa} B_c^{\lambda} B_b^{\kappa},$$

$$(1.4)\qquad H_{cb}^{\cdot\cdot\kappa} = D_c B_b^{\kappa} \equiv \partial_c B_b^{\kappa} + \Gamma_{\mu\lambda}^{\kappa} B_c^{\mu} B_b^{\lambda} - {}'\Gamma_{cb}^{a} B_a^{\kappa},$$

where D denotes the generalized covariant differentiation with respect to V_m in V_n; and $\Gamma_{\mu\lambda}^{\kappa}$ and $'\Gamma_{cb}^{a}$ are, respectively, the Christoffel symbols of the second kind for V_n and V_m.

By definition a V_m in V_n is said to be *totally semi-umbilical*[3] in V_n if a vector v_κ exists such that

$$(1.5)\qquad v_\kappa H_{cb}^{\cdot\cdot\kappa} = {}'g_{cb}$$

is satisfied at every point of V_m. In particular, this condition is evidently fulfilled when $H_{cb}^{\cdot\cdot\kappa}$ has the form $H_{cb}^{\cdot\cdot\kappa} = {}'g_{cb} n^\kappa$, n^κ being a certain vector; in this case we call V_m *totally umbilical* in V_n.

In what follows we shall consider the subspaces V_m: $x^p=$const. in a V_n with fundamental tensor of the form

$$(1.6)\qquad g_{\lambda\kappa} = \begin{pmatrix} g_{cb} & 0 \\ 0 & g_{qp} \end{pmatrix}, \qquad \begin{array}{l} a, b, \cdots, f = 1, \cdots, m, \\ p, q, \cdots, s = m+1, \cdots, n. \end{array}$$

Received by the editors, April 22, 1942.

[1] The author is a Chinese Ying-Keng Funds student. He wishes to thank Professor D. J. Struik for the conversations they had from time to time during his stay at Cambridge, Massachusetts.

[2] For the theory of subspaces V_m in a Riemannian n-space V_n, see Schouten-Struik, *Einführung in der neuern Methoden der Differentialgeometrie* II, Groningen, 1938 chap. 3.

[3] D. Perepelkine, *Sur la courbure et les espaces normaux d'une V_m dans R_n*, Rec. Math. (Mat. Sbornik) N.S. vol. 42 (1935) pp. 81–100.

The two families of subspaces V_m: $x^p = \text{const.}$ and V_{n-m}: $x^a = \text{const.}$ are called *completementary families* of subspaces in V_n. Recently, Yano[4] proved that a condition for V_m to be totally umbilical in V_n is that g_{cb} be of the form[5] $g_{cb} = \sigma(x^\kappa)\bar{g}_{cb}(x^a)$. We shall obtain, among other results, a similar condition for V_m to be totally semi-umbilical in V_n.

2. **First normal complex.** Let w^a and y^a be two arbitrary vectors in V_m. Then the vector $y^c w^b H_{cb}^{\cdot\cdot\kappa}$ spans the first normal complex of V_m in V_n, whose dimensionality m_1 is therefore equal to the rank of the matrix $[H_{cb}^{\cdot\cdot\kappa}]$. In this matrix, as well as in every matrix appearing hereafter, κ or p indicates the column and the combination of $b, c, \cdots$ the row.

Now for the subspaces V_m: $x^p = \text{const.}$ in a V_n with fundamental tensor (1.6), we have

$$B_a^\kappa = \frac{\partial x^\kappa}{\partial x^a} = \delta_a^\kappa, \qquad 'g_{cb} = g_{cb}, \tag{2.1}$$

$$H_{cb}^{\cdot\cdot a} = \Gamma_{cb}^a - '\Gamma_{cb}^a, \qquad H_{cb}^{\cdot\cdot p} = \Gamma_{cb}^p. \tag{2.2}$$

But from (1.6) and the definition of the Christoffel symbols of the second kind

$$\Gamma_{\mu\lambda}^\kappa = (1/2)g^{\kappa\nu}(\partial_\mu g_{\nu\lambda} + \partial_\lambda g_{\nu\mu} - \partial_\nu g_{\mu\lambda})$$

it follows at once that

$$\begin{aligned} \Gamma_{cb}^a &= '\Gamma_{cb}^a, & \Gamma_{cb}^p &= -(1/2)g^{pq}\partial_q g_{cb}, \\ \Gamma_{cq}^a &= (1/2)g^{ab}\partial_q g_{cb}, & \Gamma_{cr}^p &= (1/2)g^{pq}\partial_c g_{rq}. \end{aligned} \tag{2.3}$$

Thus (2.2) become

$$H_{cb}^{\cdot\cdot a} = 0, \qquad H_{cb}^{\cdot\cdot p} = -(1/2)g^{pq}\partial_q g_{cb}. \tag{2.4}$$

And therefore the dimensionality m_1 of the first normal complex of V_m in V_n is equal to the rank of the matrix $[g^{pq}\partial_q g_{cb}]$. Since $\text{Det}(g^{pq}) \neq 0$, m_1 is also the rank of the matrix $[\partial_p g_{cb}]$. Hence[6] there

[4] K. Yano, *Conformally separable quadratic differential forms*, Proc. Imp. Acad. Tokyo vol. 16 (1940) pp. 83–86. For $n = m+1$ see L. P. Eisenhart, *Riemannian geometry*, Princeton, 1926 p. 182.

[5] Throughout this paper we denote by ρ, σ, θ, ϕ scalar functions of x^κ.

[6] See, for example, T. Levi-Civita, *The absolute differential calculus*, London, 1927 pp. 9–12.

exist m_1 components of g_{cb}, which are functionally independent with regard to x^p, such that each component of g_{cb} is expressible in terms of them and x^a. Conversely, it is evident from (2.4) that if g_{cb} has this property, the first normal complex of V_m in V_n is of dimension m_1. Hence we have this theorem.[7]

THEOREM 2.1. *The first normal complex of the subspaces $x^p = const.$ in a V_n with fundamental tensor* (1.6) *is of dimension m_1, if and only if the matrix $[\partial_p g_{cb}]$ is of rank m_1, that is if g_{cb} is of the form $g_{cb} = g_{cb}(x^a, \rho_1, \cdots, \rho_{m_1})$, where the ρ's are m_1 functions of x^κ which are functionally independent with regard to x^p.*

Now it follows from (2.4) and Theorem 2.1 that the components of the vector $y^c w^b H_{cb}^{\cdot\cdot\kappa}$, which spans the first normal complex, are

$$\overset{c}{y}\overset{b}{w} H_{cb}^{\cdot\cdot a} = 0,$$

$$\overset{c}{y}\overset{b}{w} H_{cb}^{\cdot\cdot p} = -(1/2)\overset{c}{y}\overset{b}{w} g^{pq}\left(\frac{\partial g_{cb}}{\partial \rho_1}\partial_q \rho_1 + \cdots + \frac{\partial g_{cb}}{\partial \rho_{m_1}}\partial_q \rho_{m_1}\right). \tag{2.5}$$

To see the implication of these equations let us consider a certain fixed V_{n-m}: $x^a = x_0^a$. Each V_m of the family $x^p = $const. has a point in common with V_{n-m}, at which the first normal complex of V_m lies in the tangent space of V_{n-m}. Equations (2.5) then show that these first normal complexes are orthogonal to the subspaces

$$\rho_1(\overset{a}{x_0}, \overset{p}{x}) = \text{const.}, \cdots, \rho_{m_1}(\overset{a}{x_0}, \overset{p}{x}) = \text{const.}$$

of V_{n-m}.

THEOREM 2.2. *If a V_n admits two complementary families of V_m and V_{n-m}, then the first normal complexes, dimensionality m_1, of V_m at points of any fixed V_{n-m} are orthogonal to a family of subspaces V_{n-m-m_1} in V_{n-m}.*

The condition for V_m to be minimal in V_n is $'g^{cb}H_{cb}^{\cdot\cdot\kappa} = 0$, which, by (2.1) and (2.4), can be written $g^{cb}\partial_p g_{cb} = 0$, that is, ∂_p Det $(g_{cb}) = 0$. Hence this theorem follows.[8]

THEOREM 2.3. *If a V_n admits two complementary families of V_m and V_{n-m}, a necessary and sufficient condition for V_m to be minimal in V_n is that V_{n-m} determine a correspondence between them which preserves volume.*

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[7] For $m_1 = 0$, see Eisenhart, loc. cit. p. 186 Example 13.

[8] For $n = m+1$, see Eisenhart, loc. cit. p. 179.

3. **Totally semi-umbilical** V_m. According to (1.5), (2.1) and (2.4), the condition for $x^p=\text{const.}$ to be totally semi-umbilical in V_n is that a vector v^p in V_{n-m} exists such that $v^p\partial_p g_{cb}=g_{cb}$, that is,

$$v^p\partial_p \log g_{cb} = 1. \tag{3.1}$$

From this it follows that

$$v^p\partial_p \log (g_{cb}/g_{ed}) = 0. \tag{3.2}$$

Now if the first normal complex of V_m in V_n is of dimension m_1, then by Theorem 2.1 g_{cb} is of the form

$$g_{cb} = g_{cb}(x^a, \rho_1, \cdots, \rho_{m_1}). \tag{3.3}$$

Consequently, (3.2) gives

$$\theta_1 \frac{\partial}{\partial \rho_1} \log (g_{cb}/g_{ed}) + \cdots + \theta_{m_1} \frac{\partial}{\partial \rho_{m_1}} \log (g_{cb}/g_{ed}) = 0, \tag{3.4}$$

where

$$\theta_1 = v^p\partial_p\rho_1, \cdots, \theta_{m_1} = v^p\partial_p\rho_{m_1}. \tag{3.5}$$

Conversely, let m_1 functions $\theta_1, \cdots, \theta_{m_1}$ exist satisfying (3.4). Then since $\rho_1, \cdots, \rho_{m_1}$ are independent with regard to x^p, the matrix $[\partial_p\rho_1, \cdots, \partial_p\rho_{m_1}]$ is of rank m_1. Therefore, the system of linear equations (3.5) has solutions for v^p; that is, v^p exist satisfying (3.2) and also (3.1). Hence, when (3.3) is true, (3.4) is a necessary and sufficient condition for V_m to be totally semi-umbilical in V_n.

On the other hand, by a well known theorem[9] on the essential parameters of a set of functions, equation (3.4) is also the condition that there exist m_1-1 functions $\sigma_1, \cdots, \sigma_{m_1-1}$ of x^a and the ρ's (and therefore of x^κ) such that g_{cb}/g_{ed} is expressible in terms of them and x^a; that is, that g_{cb} is of the form

$$g_{cb} = \sigma_{m_1}\bar{g}_{cb}(x^a, \sigma_1, \cdots, \sigma_{m_1-1}). \tag{3.6}$$

It is seen that $\bar{g}_{cb}$ cannot be expressed in terms of x^a and less than m_1-1 independent (with regard to x^p) functions σ's; otherwise, g_{cb} would be expressible in terms of x^a and less than m_1 functions, and consequently by Theorem 2.1, the first normal complex of V_m in V_n would be of dimension less than m_1.

THEOREM 3.1. *In a* V_n *with fundamental tensor* (1.6), *each of the subspaces* $x^p=$*const.*, *whose first normal complexes are of dimension*

[9] See, for example, L. P. Eisenhart, *Continuous groups of transformation*, Princeton, 1933, p. 9.

m_1, is totally semi-umbilical in V_n, if and only if g_{cb} is of the form

$$g_{cb} = \sigma_{m_1}\bar{g}_{cb}(x^a, \sigma_1, \cdots, \sigma_{m_1-1}),$$

where the σ's are m_1 functions of x^κ which are independent with regard to x^p.

For $m_1=1$, we have Yano's result quoted in §1.

4. **Normal complexes of higher order.** We now return to the end of §2 and consider the matrix

$$\begin{pmatrix} B_a^{\kappa} \\ H_{cb}^{\cdot\cdot\kappa} \\ D_f H_{ed}^{\cdot\cdot\kappa} \end{pmatrix}. \tag{4.1}$$

Let m_1 and m_2 be, respectively, the dimensionalities of the first and second normal complexes of V_m in V_n, then the rank of the above matrix is $m+m_1+m_2$. Taking account of (2.3), (2.4) and

$$D_f H_{ed}^{\cdot\cdot\kappa} = \partial_f H_{ed}^{\cdot\cdot\kappa} + \Gamma_{\mu\lambda}^{\kappa} H_{ed}^{\cdot\cdot\lambda} B_f^{\mu} - {}'\Gamma_{fd}^{c} H_{ec}^{\cdot\cdot\kappa} - {}'\Gamma_{fe}^{c} H_{cd}^{\cdot\cdot\kappa}$$

we can easily prove from (4.1) that the following matrices are all of rank m_1+m_2:

$$\begin{pmatrix} H_{cb}^{\cdot\cdot p} \\ D_f H_{ed}^{\cdot\cdot p} \end{pmatrix}, \quad \begin{pmatrix} H_{cb}^{\cdot\cdot p} \\ \partial_f H_{ed}^{\cdot\cdot p} + \Gamma_{fq}^{p} H_{ed}^{\cdot\cdot q} \end{pmatrix}, \quad \begin{pmatrix} g^{pq}\partial_q g_{cb} \\ g^{pq}\partial_f\partial_q g_{ed} + (1/2)(\partial_f g^{pq})\partial_q g_{ed} \end{pmatrix},$$

$$\begin{pmatrix} \partial_p g_{cb} \\ \partial_f \partial_p g_{ed} + (1/2) g_{pq}(\partial_f g^{qr})\partial_r g_{ed} \end{pmatrix}. \tag{4.2}$$

The last matrix shows that, unlike m_1, the dimensionality m_2 of the second normal complex of V_m in V_n depends not only on the nature of g_{cb} but also on that of g_{qp}.

If $g_{qp}=g_{qp}(x^r)$, that is, if the complementary V_{n-m} are totally geodesic in V_n (cf. Theorem 2.1 for $m_1=0$), the matrix (4.2) reduces to

$$\begin{pmatrix} \partial_p g_{cb} \\ \partial_p \partial_f g_{ed} \end{pmatrix}.$$

This matrix is of rank m_1+m_2, and therefore g_{cb}, $\partial_f g_{ed}$ can be expressed in terms of x^a and m_1+m_2 (but not less) functions of x^κ which are independent with regard to x^p. But

$$\begin{aligned} g_{cb} &= g_{cb}(x^a, \rho_1, \cdots, \rho_{m_1}), \\ \partial_f g_{ed} &= \phi_0 + \phi_1 \partial_f \rho_1 + \cdots + \phi_{m_1}\partial_f \rho_{m_1}, \end{aligned} \tag{4.3}$$

where the ϕ's are some functions of x^a and the ρ's. Therefore the first and second normal complexes of V_m in V_n are of dimension m_1 and m_2, if and only if (4.3) is true and $\partial_f\rho_1, \cdots, \partial_f\rho_{m_1}$ are expressible in terms of $x^a, \rho_1, \cdots, \rho_{m_1}$, and m_2 other functions $\rho_{m_1+1}, \cdots, \rho_{m_1+m_2}$, which, together with $\rho_1, \cdots, \rho_{m_1}$, form m_1+m_2 functions independent with regard to x^p.

This being the case, we have

$$\partial_p g_{cb} = \frac{\partial g_{cb}}{\partial\rho_1}\partial_p\rho_1 + \cdots + \frac{\partial g_{cb}}{\partial\rho_{m_1}}\partial_p\rho_{m_1},$$

$$\partial_p\partial_f g_{ed} = \frac{\partial\partial_f g_{ed}}{\partial\rho_1}\partial_p\rho_1 + \cdots + \frac{\partial\partial_f g_{ed}}{\partial\rho_{m_1+m_2}}\partial_p\rho_{m_1+m_2}.$$

But if w^a, y^a, z^a are three arbitrary vectors in V_m, the vectors $y^c w^b H_{cb}^{\cdot\cdot p}$ and $z^f y^e w^d D_f H_{ed}^{\cdot\cdot p}$ span the first two normal complexes of V_m in V_n. Therefore by an argument similar to that which led to Theorem 2.2, we conclude that the first two normal complexes of V_m at points of a fixed V_{n-m} are orthogonal to a family of $V_{n-m-m_1-m_2}$ in V_{n-m}.

The above result can easily be extended to cover the normal complexes of higher order of V_m in V_n; indeed we have the following two theorems.

THEOREM 4.1. *In a V_n with fundamental tensor*

$$g_{\lambda\kappa} = \begin{pmatrix} g_{cb}(x^\kappa) & 0 \\ 0 & g_{qp}(x^r) \end{pmatrix},$$

the normal complexes of the subspaces $x^p = const.$ are of dimension $m_1, m_2, \cdots$ if and only if the matrices

$$\begin{pmatrix} \partial_p g_{cb} \end{pmatrix}, \begin{pmatrix} \partial_p g_{cb} \\ \partial_p\partial_f g_{ed} \end{pmatrix}, \cdots$$

are of ranks $m_1, m_1+m_2, \cdots$, respectively.

THEOREM 4.2. *If a V_n admits two families of complementary V_m and V_{n-m} and if V_{n-m} are totally geodesic in V_n, then the first l ($l=1, 2, \cdots$) normal complexes of V_m at points of any fixed V_{n-m} are orthogonal to a family of $V_{n-m-m_1-m_2\cdots-m_l}$ in V_{n-m}.*

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