BOOK REVIEWS

Sur les ensembles de distances des ensembles de points d'un espace Euclidien. (Mémoires de l'Université de Neuchatel, vol. 13.) By Sophie Piccard. L'Université de Neuchatel, 1939. 212 pp.

Let A denote a set of points on the real number-line R. The set of all number $|a-a'|$, for any a and a' in A, is the set of distances of the set A, and is denoted by $D(A)$.

A connection between A and $D(A)$ arises naturally in the following way. Let r be a real number, and let $x \rightarrow y = x + r$ be a translation of R carrying x into y. Let this translation carry A into $A(r)$. A number z will be in both A and $A(r)$ if and only if z is in A and also at a directed distance r from some point of A. Thus the intersection $A \cdot A(r)$ contains exactly one point for each distinct unordered pair (a, a') of elements of A such that $|a-a'| = |r|$. It is not surprising, then, to find many theorems based on the cardinal number of those r for which the cardinal number of $A \cdot A(r)$ is restricted in some way. This particular connection is of largely auxiliary interest, however, for the main purpose of the present memoir is to examine for their own sake the relations between properties of A and $D(A)$. The results are so numerous and so detailed that we are obliged merely to cite a few typical items from each of the four chapters. Definitions and notation are as follows: A is congruent to B ($A \cong B$) when and only when a real number k can be found such that one of the transformations $x \rightarrow y = \pm x + k$ of R carries A into B. The complement of A in R is denoted by CA. The complement of $D(A)$ in the set $0 \leqq x < \infty$ is denoted by $D'(A)$ (reviewer's notation); since 0 is in $D(A)$ naturally, any member of $D'(A)$ is positive. If A and B are two sets, $D(A, B)$ is the set of all numbers $|a-b|$ for any a in A and b in B.

The most important material in the first chapter is a summary of general theorems, due mainly to Sierpinski, whose acknowledged influence appears clearly throughout the book. We learn, for instance, that, if A is open, an F_δ, a denumerable G_δ, Borel-measurable and with $A \cdot A(r)$ at most denumerable for each r, analytic, or of the second Baire category, then $D(A)$ is, respectively, a G_δ, an F_δ, a $G_{\delta\sigma}$, Borel-measurable, analytic, or of the second Baire category. A short proof is given of Steinhaus' important theorem that, if A has positive measure, then $D(A)$ contains an interval $0 \leqq x < k$ for some k. Finally, a partial answer is given to the perfectly natural question: Evidently $A \cong B$ implies $D(A) = D(B)$; when does $D(A) = D(B)$ imply $A \cong B$? If A and B are finite, it is necessary and sufficient to assume that, for each r, $A \cdot A(r)$ and $B \cdot B(r)$ each contains at most one element; if A and B are infinite, this assumption is insufficient.

In the second chapter, the set $\Delta(A) = D'(A) \cdot D'(CA)$ is of great importance. If $\Delta(A)$ is empty, one of its factors is empty. $\Delta(A)$ is not empty when and only when $A \cong CA$. The second chapter investigates $D(A)$ mainly on this hypothesis, which entails $D(A) = D(CA)$ and $\Delta(A) = D'(A) = D'(CA)$. Under these circumstances, $A(\pm d) = CA$ (and $(CA)(\mp d) = A$) if and only if d is in $D'(A)$. Further, if $p > 0$ and $k_1, \cdots, k_p$ are any integers, $m_1, \cdots, m_p$ are any members of $D'(A)$, and $r = k_1 m_1 + \cdots + k_p m_p$, then $|r|$ is in $D'(A)$ if and only if the integer $k_1 + \cdots + k_p$ is odd. $D'(A)$ is a proper subset of that one of A and CA which does not contain zero. If $D(A)$ contains an interval, $D'(A)$ contains no decreasing sequence (and hence is denumerable). If $\Delta(A)$ contains two incommensurable numbers, then $\Delta(A)$ contains a decreasing sequence, and is everywhere dense on the interval $0 \leqq x < \infty$, while $A \cong CA$ is everywhere dense on R. Many other results deal with measurability and the Baire categories.

The next chapter is devoted to perfect sets. If A is closed and bounded, or dense-in-itself, or perfect and bounded, then $D(A)$ is, respectively, closed, dense-in-itself, or perfect. A perfect set P is said to be of the first kind (Mirimanoff) if it arises from an interval $a \leqq x \leqq b$ by a process generalized in a specific way from the construction of the Cantor ternary set. If P is of the first kind, $D(P)$ is the interval $0 \leqq x \leqq b - a$. If P and Q are of the first kind, $D(P, Q)$ has positive Lebesgue-measure mes $D(P, Q)$. There exist bounded perfect sets P and Q such that mes $D(P) = 0$ and mes $D(Q) = 0$, while mes $D(P, Q) > 0$; there exist bounded perfect sets P and Q such that mes $D(P) > 0$ and mes $D(Q) > 0$, while mes $D(P, Q) = 0$. If mes $D(A) = 0$, then mes $D(A, A(r)) = 0$ for any real r. The remainder of this chapter contains a plethora of specific information on $D(A)$ and $D(A, B)$, where A and B consist of those numbers expressible in special ways in systems of notation with assigned bases. Presumably as an example, the base 4 receives extraordinarily searching attention.

The final chapter studies conditions on a subset D of the interval $0 \leqq x < \infty$ in order that there may exist an A for which $D = D(A)$. If D is finite, it is necessary and sufficient that some subset of D, taken in increasing order, shall form a sequence S such that D coincides with the sums of the gapless subsequences of S. (It may be necessary to repeat elements of D when forming S. Example: $A = \{0, a, 2a\} = D(A)$ with $a > 0$, $S = 0, a, a$.) Results which increase rapidly in complication are obtained directly when D has not more than six elements, or when $A \cdot A(r)$ or $D \cdot D(r)$ are suitably restricted. With conditions on A, the criterion for finite D generalizes to infinite D. If $D(A)$ and (hence) A are denumerably (nondenumerably) infinite, then $D(A) \cdot (D(A))(r)$ is necessarily denumerably (nondenumerably) infinite for denumerably (nondenumerably) many r's. If D' is of measure zero or of the first Baire category, then the continuum-hypothesis implies the existence of an A for which $D=D(A)$. Finally, many properties of D are enumerated which exclude the existence of the required A.

Aside from occasional use of transfinite induction and the continuum-hypothesis, the proofs may fairly be called elementary. On the other hand, they are often quite intricate, simple results requiring the examination of a myriad cases. Pushed with patient energy, this study has yielded a remarkable amount of detailed information, from which one may form a definite idea of the difficulties and rewards to be met in the directions here pursued. One also finds general results of great interest, which do not claim to be complete. Many problems are explicitly proposed, and many others at once suggest themselves. For this solid progress we are very substantially indebted to the author, who doubtless shares the hope that a still deeper analysis—possibly along somewhat different lines—may presently yield a more satisfying theory.

Remarks: 1. Considerable data are given on sets A in higher dimensional spaces, but such results have been ignored here for the sake of brevity.

2. Though the argument is apparently not vitiated by its omission in the text (for reasons which will not escape a reader), the reviewer feels compelled to record here the fact that the property P (p. 54) is not invariant under translation, is accordingly peculiar to the position of a set on the line, and may not be shared by a set congruent to a set which has it.

3. One should take care, on p. 39, line 19, to read "$D(A)+D(CA)$" instead of "$D(A)$" (see Property 2, p. 46).

4. This book regrettably upholds the secretive tradition under which the index is omitted.

F. A. FICKEN

Tables of probability functions, Vol. II. New York, Work Projects Administration, 1942. 21+344 pp. $2.00.

The first volume of *Probability functions* appeared in 1941; it was reviewed in Bull. Amer. Math. Soc. vol. 48 (1942) p. 201. The present