$$5p_1 + 2p_3 + 7p_4 = m + 1, \qquad 5q_1 + 2q_3 + 7q_4 = n.$$

A solution of these equations is $p_1 = 4$, $p_2 = 3$, $p_3 = 11$, $p_4 = 7$, $p_5 = 1$, $q_1 = 4$, $q_2 = 3$, $q_3 = 11$, $q_4 = 7$, $q_5 = 3$. Hence a solution of (12) is $x = \alpha s^4 t^4$, $y = \beta s^3 t^3$, $u = \lambda s^{11} t^{11}$, $v = \mu s^7 t^7$, $w = \nu s t^3$ where $s = a\alpha^3 \lambda^7 \nu + b\beta^4 \mu^{11} \nu$, $t = c\alpha^5 \lambda^2 \mu^7$.

If $x = x'$, $y = y'$, $u = u'$, $v = v'$, $w = w'$ is a given solution of (12) and the choice $\alpha = x'$, $\beta = y'$, $\lambda = u'$, $\mu = v'$, $\nu = w'$ is made then $s = t$ and the solution becomes $x = x' t^8$, $y = y' t^6$, $u = u' t^{22}$, $v = v' t^{14}$, $w = w' t^4$ which is equivalent to the given solution provided $t \neq 0$.

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VECTOR ANALOGUES OF MORERA'S THEOREM

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Let the vector

$$\mathbf{X} \equiv \mathbf{X}(x_1, x_2, x_3) \equiv \mathbf{X}(x) \equiv X_1 \mathbf{i} + X_2 \mathbf{j} + X_3 \mathbf{k}$$

be defined and continuous in the domain (non-null connected open set) D. Consider the mean-value vector

$$(1) \qquad \mathbf{X}^{(\rho)}(x) \equiv \frac{1}{|V_\rho|} \int_{V_\rho} \mathbf{X}(x + \xi) dV,$$

where V_ρ denotes the sphere

$$\overset{2}{\xi_1} + \overset{2}{\xi_2} + \overset{2}{\xi_3} < \overset{2}{\rho},$$

and $|V_\rho|$ its volume,

$$|V_\rho| \equiv 4\pi\rho^3/3.$$

The vector (1) can be defined thus for only a part D_ρ of D, but this is of no consequence since ρ is arbitrarily small.

Since $\mathbf{X}(x)$ is continuous, it follows that $\mathbf{X}^{(\rho)}(x)$ has continuous partial derivatives of the first order; these are given by

$$(2) \qquad \frac{\partial}{\partial x_p} \mathbf{X}^{(\rho)}(x) = \frac{1}{|V_\rho|} \int_{S_\rho} \mathbf{X}(x + \rho\alpha)\alpha_p d\sigma,$$

where S_ρ denotes the surface of V_ρ and $\alpha_1, \alpha_2, \alpha_3$ are the components of the unit vector along the outer normal to S_ρ.

Mean-value functions are particularly useful in giving sufficiency proofs involving lessening of differentiability conditions. We shall use mean-value vectors to establish two generalizations of Morera's theorem to vector functions.

A vector $X(x)$ having continuous partial derivatives of the first order in the finite domain D has been called[1] a *Newtonian vector* provided it is both irrotational and solenoidal; that is, provided its curl and its divergence both vanish identically in D:

$$\text{curl } X \equiv \nabla \times X \equiv \left(\frac{\partial X_3}{\partial x_2} - \frac{\partial X_2}{\partial x_3}\right)i + \left(\frac{\partial X_1}{\partial x_3} - \frac{\partial X_3}{\partial x_1}\right)j$$

(3)

$$+ \left(\frac{\partial X_2}{\partial x_1} - \frac{\partial X_1}{\partial x_2}\right)k = 0;$$

(4) $$\text{div } X \equiv \nabla \cdot X \equiv \frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \frac{\partial X_3}{\partial x_3} = 0.$$

It follows readily[2] that every Newtonian vector is a harmonic vector.

For a vector of components $X_1(x_1, x_2)$, $X_2(x_1, x_2)$, equations analogous to (3) and (4) are the Cauchy-Riemann differential equations

$$\frac{\partial X_2}{\partial x_1} = \frac{\partial X_1}{\partial x_2}, \qquad \frac{\partial X_2}{\partial x_2} = -\frac{\partial X_1}{\partial x_1}$$

for the function

$$f(z) \equiv X_2(x_1, x_2) + iX_1(x_1, x_2), \qquad z \equiv x_1 + ix_2.$$

This fact is taken as justification for considering Newtonian vectors as a generalization of analytic functions of a complex variable.

Let the vector $X(x)$ have continuous partial derivatives of the first order in D. By Stokes' theorem, $X(x)$ is irrotational if and only if for each reducible[3] closed curve C in D,

(5) $$\int_C X \cdot dR = 0,$$

where R is the vector from the origin to a moving point on C. By

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[1] D. G. Fulton and G. Y. Rainich, *Generalizations to higher dimensions of the Cauchy integral formula*, American Journal of Mathematics, vol. 54 (1932), pp. 235–241.

[2] See, for instance, H. B. Phillips, *Vector Analysis*, New York, 1933, pp. 149, 153.

[3] A closed curve (or surface) in D is reducible provided it is rectifiable (or of finite area) and can be shrunk continuously in D to a point of D. The forms of Stokes' theorem and Green's theorem referred to in this paragraph may be found in H. B. Phillips, loc. cit., pp. 62, 68, 72.

Green's theorem, $X(x)$ is irrotational if and only if for each reducible closed surface S in D,

(6) $$\int_S n \times X d\sigma = 0,$$

where n is the unit vector along the outer normal to S; and again by Green's theorem, $X(x)$ is solenoidal if and only if for each reducible closed surface S in D,

(7) $$\int_S n \cdot X d\sigma = 0.$$

It follows that Newtonian vectors are characterized by (5) and (7), and also by (6) and (7); but (cf. Morera's theorem) *the assumption that $X(x)$ has continuous partial derivatives of the first order now is redundant,*[4] as we shall show. We shall further lessen the sufficiency conditions by replacing the integral conditions (5), (6) and (7) by weaker local conditions which are implied by (5), (6) and (7), respectively.

We shall denote by $S(x, r)$ the sphere with center (x) and radius r, and by $C_1(x, r)$, $C_2(x, r)$, $C_3(x, r)$ the circles with center (x) and radius r in planes perpendicular to the x_1, x_2, x_3 axes, respectively. Let

$$C(x, r) \equiv C_1(x, r)i + C_2(x, r)j + C_3(x, r)k,$$

and let $o(r^t)$ denote a function (not always the same function) of r such that

$$\lim_{r \to 0} \frac{o(r^t)}{r^t} = 0.$$

THEOREM 1. *Let the vector $X(x)$ be continuous in the finite domain D. If*

(8) $$\int_{C(x,r)} X \cdot dR = o(r^2)$$

and

(9) $$\int_{S(x,r)} n \cdot X d\sigma = o(r^3)$$

uniformly in each sphere in D, then $X(x)$ is a Newtonian vector.

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[4] See H. B. Phillips, loc. cit., pp. 177, 178, for a proof of the result that if the continuous vector function $X(x)$ satisfies (5) and (7), then $X(x)$ is a Newtonian vector.

PROOF. Since (8) and (9) hold uniformly in each sphere in D, we have

$$\frac{1}{|V_\rho|} \int_{V_\rho} \left(\int_{C(x,r)} \boldsymbol{X} \cdot d\boldsymbol{R} \right) dV = o(r^2)$$

and

$$\frac{1}{|V_\rho|} \int_{V_\rho} \left(\int_{S(x,r)} \boldsymbol{n} \cdot \boldsymbol{X} d\sigma \right) dV = o(r^3)$$

in D_ρ. Interchanging the order of integration, we obtain

$$\text{(10)} \qquad \int_{C(x,r)} \boldsymbol{X}^{(\rho)} \cdot d\boldsymbol{R} = o(r^2)$$

and

$$\text{(11)} \qquad \int_{S(x,r)} \boldsymbol{n} \cdot \boldsymbol{X}^{(\rho)} d\sigma = o(r^3).$$

Since $\boldsymbol{X}^{(\rho)}(x)$ has continuous partial derivatives of the first order in D_ρ, about any point $(x^{(0)})$ in D_ρ we have the finite Taylor development

$$\boldsymbol{X}^{(\rho)}(x) = \boldsymbol{X}^{(\rho)}(x^{(0)}) + \frac{\partial \boldsymbol{X}^{(\rho)}}{\partial x_1} (x_1 - x_1^{(0)}) + \frac{\partial \boldsymbol{X}^{(\rho)}}{\partial x_2} (x_2 - x_2^{(0)})$$

$$+ \frac{\partial \boldsymbol{X}^{(\rho)}}{\partial x_3} (x_3 - x_3^{(0)}) + o(r),$$

where the partial derivatives are evaluated at $(x^{(0)})$. Computations yield

$$\text{(12)} \qquad \int_{C(x^{(0)},r)} \boldsymbol{X}^{(\rho)} \cdot d\boldsymbol{R} = \pi r^2 \, \text{curl} \, \boldsymbol{X}^{(\rho)}(x^{(0)}) + o(r^2)$$

and

$$\text{(13)} \qquad \int_{S(x^{(0)},r)} \boldsymbol{n} \cdot \boldsymbol{X}^{(\rho)} d\sigma = (4/3)\pi r^3 \, \text{div} \, \boldsymbol{X}^{(\rho)}(x^{(0)}) + o(r^3).$$

Now (10)–(13) give

$$\text{curl} \, \boldsymbol{X}^{(\rho)}(x) = 0$$

and

$$\text{div} \, \boldsymbol{X}^{(\rho)}(x) = 0$$

throughout D_ρ; that is, $\boldsymbol{X}^{(\rho)}(x)$ is a Newtonian vector. Hence $\boldsymbol{X}(x)$, the uniform limit of $\boldsymbol{X}^{(\rho)}(x)$ as $\rho \to 0$, is a Newtonian vector.

THEOREM 2. *Let the vector $\boldsymbol{X}(x)$ be continuous in the finite domain D. If*

$$\int_{S(x,r)} n \times \boldsymbol{X} d\sigma = o(r^3)$$

and

$$\int_{S(x,r)} n \cdot \boldsymbol{X} d\sigma = o(r^3)$$

uniformly in each sphere in D, then $\boldsymbol{X}(x)$ is a Newtonian vector.

PROOF. The proof is analogous to that of Theorem 1, with

$$\int_{S(x^{(0)},r)} n \times \boldsymbol{X}^{(\rho)} d\sigma = (4/3)\pi r^3 \operatorname{curl} \boldsymbol{X}^{(\rho)}(x^{(0)}) + o(r^3)$$

in place of (12).

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