SOME NOTES ON AN EXPANSION THEOREM OF PALEY AND WIENER

R. J. DUFFIN AND J. J. EACHUS

Paley and Wiener[1] have formulated a criterion for a set of functions $\{g_n\}$ to be "near" a given orthonormal set $\{f_n\}$. The interest of this criterion is that it guarantees the set $\{g_n\}$ to have expansion properties similar to an orthonormal set.[2] In particular, they show that the set $\{g_n\}$ approximately satisfies Parseval's formula. In the first part of this paper we show that, conversely, if a set $\{g_n\}$ approximately satisfies Parseval's formula then there exists at least one orthonormal set which it is "near."

In the second part of the paper we consider sets which are on the borderline of being near a given orthonormal set.

The last part of this paper gives a simple formula for constructing sets near a given orthonormal set. As an application of this formula we obtain new properties of the so called non-harmonic Fourier series.

We shall handle these problems abstractly, using the notation of Hilbert space.[3] Subscript variables are assumed to range over all positive integers and $\sum$ shall mean a sum over all positive integers. By a finite sequence shall be meant a sequence with only a finite number of nonzero members. For application to the space L_2 the norm of a function $f(x)$ is defined in the usual way as $\|f\| = (\int_a^b |f(x)|^2 dx)^{1/2}$. A complete set which satisfies the Paley-Wiener criterion shall be termed strongly complete.

The principal novelty in the proof is the association of a linear transformation G with each set of elements $\{g_n\}$. Thus if $\{\psi_n\}$ is an orthonormal set we define $G\sum a_n\psi_n = \sum a_n g_n$ for every finite sequence of constants $\{a_n\}$. The norm of G is the limit superior of $\|Gx\|$ for elements x such that $\|x\| = 1$. With this definition of norm the aggregate of bounded linear transformations clearly forms a normed linear

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[1] R. E. A. C. Paley and N. Wiener, *Fourier Transforms in the Complex Domain*, American Mathematical Society Colloquium Publications, vol. 19, 1934, p. 100.

[2] R. P. Boas, Jr., Journal of the London Mathematical Society, vol. 14 (1939), p. 242; Duke Mathematical Journal, vol. 6 (1940), p. 148; American Journal of Mathematics, vol. 63 (1941), p. 361.

[3] Because of the difficulty of finding adequate references to non-separable Hilbert space we confine ourselves to separable space. However, our theorems remain true for non-separable Hilbert space provided the range of subscript variable is redefined.

space. It is in this space that the concept of one sequence of functions being "near" another is made precise; thus two points are "near" if they are within unit distance of each other.

1. **A converse theorem.** We prove the following theorem.

THEOREM A. *Suppose that the set of elements* $\{g_n\}$ *satisfies the relation*

$$(1)\qquad (1-\theta)\left(\sum |a_n|^2\right)^{1/2} \leqq \left\| \sum a_n g_n \right\| \leqq (1+\theta)\left(\sum |a_n|^2\right)^{1/2}$$

for every finite sequence of constants $\{a_n\}$ *and for some fixed number* θ; $0 \leqq \theta < 1$. *Then there exists an orthonormal set* $\{f_n\}$ *such that*

$$(2)\qquad \left\| \sum a_n(f_n - g_n) \right\| \leqq \theta\left(\sum |a_n|^2\right)^{1/2}.$$

PROOF. Relation (1) guarantees that the set $\{g_n\}$ is linearly independent; hence the closure of all linear combinations of the set $\{g_n\}$ is an infinite-dimensional manifold, a Hilbert space. Let $\{\psi_n\}$ be some complete orthonormal set of this Hilbert space and define the transformation G by $G\sum a_n\psi_n = \sum a_n g_n$. Thus if $y = \sum a_n \psi_n$ we may express relation (1) in the form

$$(3)\qquad (1-\theta)\|y\| \leqq \|Gy\| \leqq (1+\theta)\|y\|.$$

The following lemmas are well known:[4]

LEMMA 1. *A bounded linear transformation whose range and domain both determine Hilbert space may be factored in the form* FP *where* F *is a unitary transformation and* P *is a positive definite self-adjoint transformation.*

LEMMA 2. *If* S *is a self-adjoint transformation then*

$$\|Sx\|/\|x\|$$

and

$$|(Sx, x)|/(x, x)$$

have the same upper and lower bounds.

Clearly G satisfies the conditions of Lemma 1, so $G = FP$. From relation (3) it follows that

$$(1-\theta)\|y\| \leqq \|Py\| \leqq (1+\theta)\|y\|.$$

[4] M. H. Stone, *Linear Transformations in Hilbert Space and Their Applications to Analysis*, American Mathematical Society Colloquium Publications, vol. 15, 1932; J. von Neumann, Annals of Mathematics, (2), vol. 33 (1932), p. 308; A. Wintner, Mathematische Annalen, vol. 37 (1933), p. 257.

Moreover, this inequality clearly remains valid if y is an arbitrary element; so by Lemma 2

$$(1-\theta)(y, y) \leqq (Py, y) \leqq (1+\theta)(y, y).$$

This may be written as

$$-\theta(y, y) \leqq (Py - y, y) \leqq \theta(y, y).$$

Again using Lemma 2, we have $\|y-Py\| \leqq \theta\|y\|$. Let us define $f_n = F\psi_n$. Because F is unitary it follows that $\{f_n\}$ is an orthonormal set. Moreover,

$$\begin{aligned} \left\|\sum a_n(f_n - g_n)\right\| &= \|Fy - Gy\| \\ &= \|F(y - Py)\| \\ &= \|y - Py\| \\ &\leqq \theta\|y\| = \theta\left(\sum |a_n|^2\right)^{1/2}. \end{aligned}$$

2. **The borderline case,** $\theta = 1$. We now establish this theorem.

THEOREM B. *The set of elements* $\{g_n\}$ *and the orthonormal set* $\{f_n\}$ *satisfy*

$$\left\|\sum a_n(f_n - g_n)\right\| < \left(\sum |a_n|^2\right)^{1/2} \tag{4}$$

for every sequence of coefficients $\{a_n\}$ *such that*

$$0 < \sum |a_n|^2 < \infty.$$

Then $\{g_n\}$ *is complete if* $\{f_n\}$ *is complete.*[5]

PROOF. If $\{g_n\}$ is not complete, there exists an element z such that $(g_n, z) = 0$ for all n. We can express z in the form $z = \sum a_n f_n$, where $0 < \sum |a_n|^2 < \infty$. Thus

$$\left\|\sum a_n(f_n - g_n)\right\|^2 = \|z\|^2 - 2\Re \sum a_n(g_n, z) + \left\|\sum a_n g_n\right\|^2 \geqq \|z\|^2.$$

This contradicts (4), and the theorem is proved.

The following counterexample shows that the completeness of the set $\{g_n\{$ does not imply that the orthonormal set $\{f_n\}$ is complete. Let $\{c_n\}$ be a sequence of constants such that $c_n > c_{n+1} > 1$. Let $\{\psi_n\}$

[5] The referee has pointed out that a mean ergodic theorem may be used to prove the following generalization of Theorem B.

Let B be a reflexive Banach space and let $\{f_n\}$ *be a basis in B. Suppose* $\{g_n\}$ *is a sequence in B such that* $\|\sum a_n(f_n - g_n)\| < \|\sum a_n f_n\|$ *for all* $x = \sum a_n f_n$ *in B. Then the elements* $\{g_n\}$ *span B.*

We have been able to extend his theorem to spaces whose unit sphere has weak sequential compactness.

be a complete orthonormal set. Define $f_n = \psi_{n+1}$ and $g_n = \psi_{n+1} - \psi_n c_{n+1}/c_n$ for $n = 1, 2, \cdots$. Thus

$$\left\| \sum a_n(f_n - g_n) \right\|^2 = \sum \left| \frac{c_{n+1}}{c_n} a_n \right|^2 < \sum | a_n |^2,$$

and so relation (4) is satisfied. Obviously the set $\{f_n\}$ is not complete; however $\sum | c_n |^2 = \infty$, so the set $\{g_n\}$ is complete.

If the $<$ sign is replaced by the $\leqq$ sign in relation (4), Theorem B is certainly false because some or all of the elements $\{g_n\}$ could be zero. Nevertheless, we are able to prove an analogue of Theorem A.

THEOREM C. *Let $\{g_n\}$ be a complete set and $\{f_n\}$ an orthonormal set such that*

$$\left\| \sum a_n(f_n - g_n) \right\| \leqq \left(\sum | a_n |^2\right)^{1/2} \tag{5}$$

for every sequence of constants $\{a_n\}$ such that

$$0 < \sum | a_n |^2 < \infty. \tag{6}$$

Then

$$0 < \left\| \sum a_n g_n \right\| \leqq 2\left(\sum | a_n |^2\right)^{1/2}. \tag{7}$$

Conversely, the truth of (7) *for a complete set $\{g_n\}$ and for every sequence of constants $\{a_n\}$ satisfying* (6) *implies the existence of a* (*complete*) *orthonormal set $\{f_n\}$ satisfying* (5).

PROOF. The novelty in the first part of this theorem is the appearance of the strict inequality on the left side of (7). Suppose then, on the contrary, that $\{a_n\}$ is a sequence for which $\sum a_n g_n = 0$. Let $\sum a_n f_n = z$. Because $\{g_n\}$ is complete, there exists an element g_c such that $(g_c, z) \neq 0$. Thus, according to (5), if λ is a constant:

$$\left\| \sum a_n(f_n - g_n) + \lambda(f_c - g_c) \right\| \leqq \left\| \sum a_n f_n + \lambda f_c \right\|,$$

$$\left\| z + \lambda f_c - \lambda g_c \right\| \leqq \left\| z + \lambda f_c \right\|,$$

$$- 2\Re\lambda(g_c, z) + | \lambda |^2 \{ | g_c |^2 - 2\Re(g_c, f_c) \} \leqq 0.$$

This last inequality is clearly impossible for all values of λ. The remainder of the proof is omitted as it parallels the proof of Theorem A.

3. **The method of separation of variables.** We prove the following theorem.

THEOREM D. *Let $\{C_{nk}\}$, $n, k = 1, 2, \cdots$, be a matrix of constants such that $| C_{nk} | \leqq c_k$. Let $\{T_k\}$ be a sequence of bounded linear trans-*

formations with corresponding bounds $\{t_k\}$. *Let* $\{f_n\}$ *be a complete orthonormal set, and define*

$$g_n = f_n + \sum_{k=1}^{\infty} C_{nk} T_k f_n.$$

Then if $\sum c_k t_k < 1$, *the set* $\{g_n\}$ *is strongly complete in the sense of Paley and Wiener.*

PROOF. Let $\{a_n\}$ be an arbitrary finite sequence of constants. Then

$$\begin{aligned} \left\| \sum a_n(f_n - g_n) \right\| &= \left\| \sum_n a_n \sum_k C_{nk} T_k f_n \right\| \\ &= \left\| \sum_k T_k \sum_n C_{nk} a_n f_n \right\| \\ &\leqq \sum_k \left\| T_k \sum_n C_{nk} a_n f_n \right\| \\ &\leqq \sum_k t_k \left\| \sum_n C_{nk} a_n f_n \right\| \\ &\leqq \sum_k t_k c_k \left\| \sum_n a_n f_n \right\|. \end{aligned}$$

Thus the set $\{g_n\}$ satisfies the Paley-Wiener criterion (2) with $\theta = \sum t_k c_k < 1$.

A simple way to apply Theorem D depends upon the fact that multiplication by a bounded function is a bounded linear transformation of the space L_2. In particular we consider the sequence of functions $\{e^{i\lambda_n x}\}$, where $\{\lambda_n\}$, $n = 0, \pm 1, \pm 2, \cdots$, is a sequence of complex constants satisfying $|\lambda_n - n| \leqq L$ for some constant L. The interval under consideration is $-\pi \leqq x \leqq \pi$. We may write

$$e^{i\lambda_n x} = e^{inx} + \sum_{k=1}^{\infty} \frac{(i\lambda_n - in)^k}{k!} x^k e^{inx}.$$

Comparing with Theorem D, we have:

$$\begin{aligned} C_{nk} &= (i\lambda_n - in)^k / k!, \qquad & c_k &= L^k / k!; \\ T_k &= x^k, & t_k &= \pi^k. \end{aligned}$$

Clearly, if $L < \log 2/\pi$, the set $\{e^{i\lambda_n x}\}$ is strongly complete in the interval $(-\pi, \pi)$. (Actually the same is true in any interval of length 2π.)

The transformations $\{x^k\}$ never attain their least upper bound. This fact permits Theorem B to be employed in showing that the set $\{e^{i\lambda_n x}\}$ is complete even if $L = \log 2/\pi$. The proof parallels the proof of Theorem D.

The above results on the non-harmonic Fourier series are an extension of previous knowledge in two respects: In the first place, Paley and Wiener were forced to assume that $\{\lambda_n\}$ was a real sequence. Secondly, they[6] obtained the value $1/\pi^2 = .10+$ where we have $\log 2/\pi = .22+$. The best value for L is not known; however a theorem of Levinson[7] gives an upper limit of $1/4$.[8]

A second application of Theorem D is to furnish a proof of an analytic function expansion theorem of Boas.[9] In turn, Boas' theorem contains analytic function expansion theorems of Birkhoff, Walsh, Takenaka, G. S. Ketchum and others.

The operator in Theorem D, $\sum C_{nk} T_k$, has been assumed to be a discrete series; however the method of separation of variables is still available if we replace the series by an integral or Stieltjes integral. In particular, Cauchy's integral formula is of the right form.

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PURDUE UNIVERSITY

[6] Paley and Wiener, loc. cit., p. 113. A slightly better value than theirs has been obtained by Malin, thesis, Massachusetts Institute of Technology, 1934.

[7] N. Levinson, Annals of Mathematics, (2), vol. 37 (1936), p. 919; *Gap and Density Theorems*, American Mathematical Society Colloquium Publications, vol. 26, 1940, chap. IV.

[8] It is a curious parallelism that $\log 2/\pi$ and $1/4$ are in the same ratio as the limits of Takenaka and Schoenberg in a somewhat similar unsolved problem. For references see R. P. Boas, Jr., Proceedings of the National Academy of Sciences, vol. 26 (1940), p. 139; Transactions of this Society, vol. 48 (1940), p. 485.

[9] R. P. Boas, Jr., Transactions of this Society, vol. 48 (1940), p. 473, Theorem 3.1.