A NOTE ON THE DIOPHANTINE PROBLEM OF FINDING FOUR BIQUADRATES WHOSE SUM IS A BIQUADRATE

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The problem is to assign integers satisfying the relation

$$x^4 = y^4 + a^4 + b^4 + c^4. \tag{1}$$

Granted that the five integers have no common divisor, it follows that x and one quantity on the right (here taken as y) are not multiples of 5; and it can be shown that (1) without loss of generality can be replaced by

$$x^4 = y^4 + 5^4t^4(\alpha^4 + \beta^4 + \gamma^4). \tag{2}$$

In this expression x is an odd number prime to y; y may be odd or even; and neither is divisible by 5, as stated above.

Let $(x^4-y^4)/5^4t^4=d$; which for values satisfying (2) will be an integer.

Evidently x and y must satisfy the congruence $x^4 \equiv y^4 \pmod{5^4}$. Hence, for values of x up to any required magnitude, there are corresponding values of y;[1] and the resultant values of d can be found and tabulated.

If (2) is satisfied, then $d=\alpha^4+\beta^4+\gamma^4$; and from the elementary properties of the sum of three biquadrates it is seen that many values of d may be rejected at once: for example, all those having for the final digit 0, 4, 5, or 9; and all those incapable of representing the sum of three square numbers. The remaining values of d will have for the final digit 1, 2, 3, 6, 7, or 8. Regarding these as separate cases, it is possible to devise a numerical test for each case. As an example, consider the values of d ending with $\cdots 3$. These must have the form $80k+3$ (sum of three odd biquadrates prime to 5) or the form $80k+33$ (sum of one odd and two even biquadrates prime to 5). Choosing the latter form,[2] seventy instances are found in a tabulation of values of d based on all admissible values of $x<700$. The first ten in order of magnitude are: 164833, 195313, 198593, 3029873, 4106193, 4590753, 5086913, 5693793, 5948193, 7424753. Testing these by a method (based upon the character of k), which need not be

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[1] Solutions of the congruence in question can be read from C. J. G. Jacobi's *Canon Arithmeticus*, Berlin, 1839, pp. 230–231.

[2] This form was considered first because it includes Norrie's solution, which appears in The University of St. Andrew 500th Anniversary Memorial Volume, Edinburgh, 1911, p. 89.

described in this brief discussion, it is found that the third integer is the sum of three biquadrates, and gives Norrie's solution

$$d = 198593 = \frac{353^4 - 272^4}{15^4} = 21^4 + 8^4 + 2^4.$$

The seventh integer furnishes a second numerical solution

$$d = 5086913 = \frac{651^4 - 599^4}{10^4} = 43^4 + 34^4 + 24^4.$$

It is the intention of the writer to carry the investigation of each case as far as values of $x < 1250$ $(=2 \cdot 5^4)$. The fact that an additional solution was obtained in the first case examined would seem to give some promise that other solutions exist among the lower integers.

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ON THE INVERSION OF THE q-SERIES ASSOCIATED WITH JACOBIAN ELLIPTIC FUNCTIONS

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The elliptic functions sn (u, k), cn (u, k) and dn (u, k) may be computed from theta functions by well known methods outlined in standard texts. (See, for instance, Whittaker & Watson, *Modern Analysis*, 4th edition, p. 485.) Given the modulus k, and $k' = (1-k^2)^{1/2}$, there is associated with k, k' a function ϵ defined by

$$\epsilon = \frac{1}{2}\left[\frac{1-(k')^{1/2}}{1+(k')^{1/2}}\right] = \frac{1}{2}\,\frac{\vartheta_2(0, q^4)}{\vartheta_3(0, q^4)}\,.$$

Values of theta functions for a given parameter q can be readily computed, and the Jacobi elliptic functions turn out to be ratios of the theta functions.

The series for ϵ in terms of q is given by

$$\epsilon = \sum_{k=0}^{\infty} q^{(2k+1)^2} \Big/ 1 + 2\sum_{k=1}^{\infty} q^{4k^2}. \tag{1}$$

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[1] The above results were obtained in the course of work carried out by the Mathematical Tables Project, conducted by the Work Projects Administration for the City of New York under the sponsorship of the National Bureau of Standards.