$$\delta_1\omega = \omega^{\rho_1+1}, \text{ and } \delta_1 j = \omega^{\rho_1}v_1 j + \omega^{\rho_2}v_2 + \cdots + \omega^{\rho_z}v_z < \omega^{\rho_1}(v_1 j + 1);$$

$$\sigma(\delta_1\mu, \delta_1 j) < \sigma(\delta_1\mu, \omega^{\rho_1}(v_1 j + 1)) < \delta_1\mu + \omega^{\rho_1+1} = \delta_1\mu + \delta_1\omega.$$

By (2), $\pi(\delta^\mu, \delta^j) < \omega^{\delta_1\mu+\delta_1\omega} = (\omega^{\delta_1})^{(\mu+\omega)} \leqq \delta^{\mu+\omega} \leqq \delta^\delta$.

Hence by (1), the order type of S is less than $\pi(\omega^\delta, \delta^\delta)$. This is a contradiction since S was the segment of M^δ of order type $\pi(\omega^\delta, \delta^\delta)$.

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A CHARACTERIZATION OF ABSOLUTE NEIGHBORHOOD RETRACTS

RALPH H. FOX

By an *absolute neighborhood retract* (ANR) I mean a separable metrizable space which is a neighborhood retract of every separable metrizable space which contains it and in which it is closed. This generalization of Borsuk's original definition[1] was given by Kuratowski[2] for the purpose of enlarging the class of absolute neighborhood retracts to include certain spaces which are not compact. The space originally designated by Borsuk as absolute neighborhood retracts (or $\mathfrak{R}$-sets) will now be referred to as compact absolute neighborhood retracts. Many of the properties of compact ANR-sets hold equally for the more general ANR-sets.[3]

The Hilbert parallelotope Q, that is, the product of the closed unit interval $[0, 1]$ with itself a countable number of times is a "universal" compact ANR in the sense that[4] every compact ANR is homeomorphic to a neighborhood retract of Q. The classical theory of Borsuk makes good use of the imbedding of compact ANR-sets in Q. The problem solved here is that of finding a "universal" ANR.

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[1] Fundamenta Mathematicae, vol. 19 (1932), pp. 220–242.

[2] Fundamenta Mathematicae, vol. 24 (1935), p. 270, Footnote 1.

[3] Ibid., pp. 272, 276, and 277, and Footnote 1, p. 279 and Footnote 3. Note that Theorem 12, Fundamenta Mathematicae, vol. 19 (1932), p. 229, is not true for general ANR-sets. In fact let $A = \sum S_n$ where S_n is the plane circle of radius 2^{-n} and center $(3 \cdot 2^{-n}, 0)$; let $f(x, y) = (x, |y|)$ for $(x, y) \in A$ and let

$$f_n(x, y) = \begin{cases} (x, |y|), & \text{for } (x, y) \in A - S_n, \\ (x, y), & \text{for } (x, y) \in S_n. \end{cases}$$

Then $f_n \to f$ in A^A; f can be extended to the half-plane $\{x > 0\}$, but none of the maps f_n can. A is an ANR-set. Theorem 16, Fundamenta Mathematicae, vol. 19 (1932), p. 230, is also false for general ANR-sets.

[4] Fundamenta Mathematicae, vol. 19 (1932), p. 223.

Strictly speaking, the problem as just stated has no solution; there is no single "universal" ANR, but rather a whole class of ANR-sets which together serve in the "universal" capacity. Such a class of ANR-sets is the collection of subsets of the Hilbert parallelotope $Q\times[0, 1]$ which contains the open subset[5] $Q\times(0, 1]$ of $Q\times[0, 1]$.

THEOREM 1. *For a separable metrizable space X the following three conditions are equivalent*:

(1) *X is an* ANR-*set*;

(2) *There is a homeomorphism f of X into Q such that $f(X)\times[0]$ is a neighborhood retract of $f(X)\times[0]+Q\times(0, 1]$*;

(3) *$f(X)\times[0]$ is a neighborhood retract of $f(X)\times[0]+Q\times(0, 1]$ for every homeomorphism f of X into Q.*

(1)→(3): If f is a homeomorphism of an ANR-set X into Q then $f(X)\times[0]$ is an ANR-set. Since Q is compact, so that

$$\overline{f(X)\times[0]}\subset Q\times[0],$$

it follows that $f(X)\times[0]$ is closed in $f(X)\times[0]+Q\times(0, 1]$. Hence $f(X)\times[0]$ is a neighborhood retract of $f(X)\times[0]+Q\times(0, 1]$.

(3)→(2): Since X is separable and metrizable a homeomorphism f exists by Urysohn's theorem.[6]

(2)→(1): Let M be a separable metrizable space containing X in which X is closed and let f be a homeomorphism of X into Q. By Tietze's theorem[7] there exists a continuous function g defined on M with values in Q such that $g(x)=f(x)$ for every $x\in X$. Let M be metrized, with metric d, and let $\rho(x)=\min\{1, d(x, X)\}$ for every $x\in M$. Let $h(x)=(g(x), \rho(x))$, so that h is a continuous function defined on M with values in $f(X)\times[0]+Q\times(0, 1]$ which has the property $h(M-X)\subset Q\times(0, 1]$. Let V be a neighborhood of $f(X)\times[0]$ in $f(X)\times[0]+Q\times(0, 1]$ and let $U=h^{-1}(V)$ so that U is a neighborhood of X in M. If r is a retraction of V onto $f(X)\times[0]$ then the mapping[8] $f^{-1}\pi rh|U$, where π denotes the projection of $Q\times[0]$ onto Q, is a retraction of U onto X.

Kuratowski also gave an analogous generalization of the notion of absolute retract.[2] According to the extended definition a separable metrizable space is an *absolute retract* (AR) if it is a retract of every containing separable metrizable space in which it is closed.

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[5] The symbol $(0, 1]$ denotes the half-open interval $0<t\leqq 1$.

[6] Alexandroff and Hopf, *Topologie*, p. 81.

[7] Ibid., p. 73.

[8] If $B\subset B'$ and e is a function defined on B' then the notation $d=e|B$ means that d is the function defined on B such that $d(x)=e(x)$ for every $x\in B$.

THEOREM 1′. *For a separable metrizable space X the following three conditions are equivalent*:

(1′) *X is an* AR;

(2′) *There is a homeomorphism f of X into Q such that $f(X)\times[0]$ is a retract of $f(X)\times[0]+Q\times(0, 1]$*;

(3′) *$f(X)\times[0]$ is a retract of $f(X)\times[0]+Q\times(0, 1]$ for every homeomorphism f of X into Q.*

The proof of this theorem is an obvious modification of the preceding proof.

COROLLARY. *If C denotes the open n-cell $0<x_i<1$ $(i=1, \cdots, n)$ and D denotes the closed n-cell $0\leqq x_i\leqq 1$ $(i=1, \cdots, n)$ then any set E such that $C\subset E\subset D$ is an* AR.

By condition (2′) and a retraction of $Q\times[0, 1]$ onto $D\times[0, 1]$ it is sufficient to show that $E\times[0]$ is a retract of $E\times[0]+D\times(0, 1]$. This can be done by projecting from the point $(1/2, \cdots, 1/2, -1)$ of Euclidean $(n+1)$-space.

It may be worth noting that conditions (2) and (2′) make possible a simpler proof of the Borsuk-Kuratowski[9] theorem(s):

If W is a closed subset of a normal space Z and X is an AR-*set* (ANR-*set*) *then every continuous map of W into X can be extended to Z (to a neighborhood of W in Z).*

In fact conditions (2) and (2′) replace a theorem of Kuratowski[10] which involves infinite polyhedra.

THEOREM 2. *An* ANR *is locally contractible.*[11] *An* AR *is also contractible.*

Using (2) we can suppose that our ANR-set Y is contained in $Q\times[0]$ and that there is a retraction r of an open neighborhood V of Y in $Y+Q\times(0, 1]$ onto Y. But V is the intersection of $Y+Q\times(0, 1]$ with an open set V' of $Q\times[0, 1]$. Let $y\in Y$ and let S_ϵ denote the ϵ-sphere in $Q\times[0, 1]$ about the point y. Since r is continuous there is a $\delta>0$ such that the intersection T_δ of the δ-sphere S_δ and $Y+Q\times(0, 1]$ is contained in V', hence in V, and $r(T_\delta)\subset S_\epsilon$. Let u_t denote a contraction of S_δ to a point $p\in S_\delta\cdot(Q\times(0, 1])$ which moves points rectilinearly, so that $u_t(x)\in Q\times(0, 1]$ for every $0<t\leqq 1$ and

[9] Fundamenta Mathematicae, vol. 24 (1935), p. 275.

[10] Fundamenta Mathematicae, vol. 24 (1935), p. 266, Theorem 2.

[11] But not uniformly. See the example in Footnote 3. This theorem was proved by Borsuk, Fundamenta Mathematicae, vol. 19 (1932), p. 237 for compact ANR-sets.

$y \in Y \cdot S_\delta$. Then $ru_t | Y \cdot S_\delta$ contracts $Y \cdot S_\delta$ in $Y \cdot S_\epsilon$. The second statement is a consequence of Theorem 3′.

THEOREM 3. *A separable metrizable space X is an* ANR *if and only if for every separable metrizable space M containing X (in which X need not be closed!) there is a neighborhood U of X and a continuous function h defined on $X\times[0]+U\times(0, 1]$ with values in X such that*[8] *$h|X\times[0, 1]$ is a deformation.*[12]

Suppose X is an ANR and M a separable metrizable space containing X. We may assume that $M\subset Q$. By (2) and (3) there is an open neighborhood V' of $X\times[0]+Q\times(0, 1]$ and a retraction r of $V=V'\cdot(X\times[0]+Q\times(0, 1])$ onto $X\times[0]$. Let $\lambda(x)=d(x\times[0], Q\times[0, 1]-V')$ for every $x\in M$ and let $U=\pi(V'\cdot(Q\times[0]))$ where, as before, π denotes the projection of $Q\times[0]$ onto Q. Define for every $(x, t)\in X\times[0]+U\times(0, 1]$,

$$\begin{aligned} h(x, t) &= \pi r(x, t), && \text{when } t \leqq \lambda(x), \\ &= \pi r(x, \lambda(x)), && \text{when } t \geqq \lambda(x). \end{aligned}$$

Since λ is continuous and $\lambda(x)>0$ when $x\in U$ it follows that h is continuous.

Conversely, let U be a neighborhood of X in $M=Q$ and let h be a continuous function defined on $X\times[0]+U\times(0, 1]$ with values in X such that $h(x, 0)=x$ for every $x\in X$. Then h is a retraction of $X\times[0]+U\times(0, 1]$ onto $X\times[0]$. Furthermore $X\times[0]+U\times(0, 1]$ is a neighborhood of $X\times[0]$ in $X\times[0]+Q\times(0, 1]$.

THEOREM 3′. *A separable metrizable space X is an* AR *if and only if for any separable metrizable space M containing X there is a continuous function h defined on $X\times[0]+M\times(0, 1]$ with values in X such that*[8] *$h|X\times[0, 1]$ is a contraction.*[12]

Let X be an AR and M a separable metrizable space containing X; we may assume that $M\subset Q$. Let r be a retraction of $X\times[0]+Q\times(0, 1]$ onto $X\times[0]$. Let $p\in Q$ and let

$$h(x, t) = \pi r(tp + (1 - t)x, t)$$

for every $(x, t)\in X\times[0]+M\times(0, 1]$, where π is the projection of $Q\times[0]$ onto Q. Then h maps $X\times[0]+M\times(0, 1]$ continuously into X and $h|X\times[0, 1]$ is a contraction of X.

The converse is proved as in Theorem 3.

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[12] A *deformation* of X is a continuous mapping h of $X\times[0, 1]$ into X such that $h(x, 0)=x$ for every $x\in X$. If $h(X, 1)$ is a point then h is called a *contraction* of X.

If X is locally compact the deformation $h|X\times[0, 1]$ of Theorems 3 and 3' can be chosen in advance of M. For then there exists[13] a compact set M^* and a homeomorphism g of X into M^* such that $M^*-g(X)$ is a point. (We can suppose X not compact so that $M^*\neq g(X)$.) Let $M^*\subset Q$. The homeomorphism g can be extended[13] to a continuous mapping g^* of $\overline{X}$ into M^* by defining $g^*(\overline{X}-X)=M^*-g(X)$. The mapping g^* of $\overline{X}$ into Q can be extended, by Tietze's theorem, to a mapping k of M into Q. In the case of Theorem 3 let h be the mapping of $X\times[0]+U\times(0, 1]$ into X defined by

$$h(x, t) = g^{-1}\pi r(k(x), \min\{t, \lambda(x)\}),$$

where $U=g^{-1}\pi(V'\cdot(Q\times[0]))$. In the case of Theorem 3' let h be the mapping of $X\times[0]+M\times(0, 1]$ into X defined by

$$h(x, t) = g^{-1}\pi r(tp + (1 - t)k(x), t).$$

In both cases $h|X\times[0, 1]$ is independent of M.

If X is not locally compact it may not be possible to pick a deformation $h|X\times[0, 1]$ satisfying the conditions of Theorems 3 or 3' for all M. An example is the AR-set $\{0\leqq x\leqq 1;\ y=0\}+\sum_{n=1}^{\infty}\{x=1/n;\ 0\leqq y\leqq 1\}$.

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[13] Alexandroff and Hopf, *Topologie*, I, p. 93.