tive radius r. Let the center x_0 be the sequence $\{k_i^0\}$, and let s be chosen so large that $2^{-s-1}+2^{-s-2}+\cdots<r$. Now a point $X\equiv\{j_i\}$ of D exists such that $\lim_n f_n(X)=+\infty$, and such that $j_{s+1}>k_s^0$. If we define x_1 as $(k_1^0, k_2^0, \cdots, k_s^0, j_{s+1}, j_{s+2}, \cdots)$, then x_1 belongs to K and $\lim_n f_n(x_1)=+\infty$. Consequently x_1 cannot be a point of U_μ and this contradiction establishes U as a set of the first category.

In a similar fashion it may be shown that the set V of all x in D for which $\lim\inf_n f_n(x)>-\infty$ is likewise a set of the first category. Hence if we set $W\equiv U+V$ the theorem follows.

Finally, let $\sum u_k$ be a convergent series of complex terms for which $\sum|u_k|=+\infty$, and for this series let $\phi_n(\xi)$ $[f_n(x)]$ be defined as in (1.5) [(2.4)]. We may consider the series of real and imaginary parts in the light of Theorem 2 [Theorem 3] and thus show that the set of all ξ on I [x in D] for which we have $\lim\sup_n |\phi_n(\xi)|<\infty$ [$\lim\sup_n|f_n(x)|<\infty$] is a set of the first category.

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A FORMULA FOR THE DIRECT PRODUCT OF CROSSED PRODUCT ALGEBRAS

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1. Introduction. In this note we wish to present a uniform treatment of certain properties of crossed products. A crossed product over any field F is an algebra determined by a finite, separable, normal extension N of F, with a Galois group Γ, and a certain factor set[1] $\boldsymbol{h}$ of elements $h_{S,T}$ in N, for automorphisms S and T in Γ. The crossed product $(N, \Gamma, \boldsymbol{h})$ consists of all sums $\sum u_S z_S$, where the coefficients z_S lie in N, and the fixed elements u_S have the multiplication table

$$(1) \qquad u_S u_T = u_{ST} h_{S,T}, \qquad z u_S = u_S z^S, \qquad z \text{ in } N.$$

Let K be a normal subfield of N, corresponding to the subgroup Δ of the Galois group Γ. A factor set $\boldsymbol{g}$ in N is called *symmetric* in Δ if $g_{S,T}=g_{U,V}$ whenever SU^{-1} and TV^{-1} are in Δ.

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[1] Definitions are given in A. A. Albert, *Structure of Algebras*, American Mathematical Society Colloquium Publications, vol. 24, 1939. Theorems cited below without explicit source all refer to this work.

THEOREM 1. *A crossed product* $(N, \Gamma, \boldsymbol{h})$ *is split by a normal subfield* K *of* N *if and only if its factor set* $\boldsymbol{h}$ *is associate to a* Δ*-symmetric factor set* $\boldsymbol{g}$, *where* Δ *is the subgroup of* Γ *corresponding to* K.

Part of this result may be stated more explicitly, using the factor group Γ/Δ as the Galois group of K over F.

THEOREM 2. *A factor set* $\boldsymbol{g}$ *is* Δ*-symmetric if and only if there is a factor set* $\boldsymbol{G}$ *of elements* $G_{\sigma,\tau}$ *in* K, *where* σ, τ *are in* Γ/Δ, *such that, for* S *in the coset* σ, T *in the coset* τ,

$$g_{S,T} = G_{\sigma,\tau}. \tag{2}$$

Furthermore the corresponding crossed products are similar,

$$(N, \Gamma, \boldsymbol{g}) \sim (K, \Gamma/\Delta, \boldsymbol{G}). \tag{3}$$

An equivalent of these theorems was stated by Deuring.[2] Since they were not used in Albert's Colloquium Lectures, Deuring's somewhat obscure proof is apparently the only one available. We give here a new proof. It is based on the simple observation that the standard proof[3] of the formula

$$(N, \Gamma, \boldsymbol{g}) \times (N, \Gamma, \boldsymbol{h}) \sim (N, \Gamma, \boldsymbol{gh}) \tag{4}$$

can be extended to treat the case $(N, \Gamma, \boldsymbol{h}) \times (K, \Gamma/\Delta, \boldsymbol{G})$. From this formula we obtain the theorems above, as well as a general formula for the direct product of two crossed products built on *any* two normal fields. In a systematic treatment, this proof has the advantage that it involves practically no more trouble than the proof of the ordinary product formula (4), and includes this as a special case.

2. **Idempotents of matric subalgebras.** It seems convenient to use the following restatement of known results about possible total matric subalgebras of a simple algebra.

THEOREM 3. *Let the unity element of a simple algebra* A *be represented as a sum* $1 = e_1 + e_2 + \cdots + e_t$ *of pairwise orthogonal idempotents. Then* A *has a total matric subalgebra* M *with a basis* e_{ij}, $i, j = 1, \cdots, t$, *having the usual multiplication table,* $e_{ij}e_{jk} = e_{ik}$, $e_{ij}e_{mk} = 0$ *for* $j \neq m$, *and so constructed that* $e_i = e_{ii}$, *for* $i = 1, \cdots, t$, *if and only if* A *has for each* i *an automorphism which maps* e_1 *on* e_i.

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[2] M. Deuring, *Algebren*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 4, no. 1 (1935), pp. 62–64.

[3] Given in Albert, partly in Theorem 2.27 and partly in Theorem 5.6, and, originally, in somewhat different form, in H. Hasse, *Theory of cyclic algebras over an algebraic number field*, Transactions of this Society, vol. 34 (1932), pp. 191–194.

PROOF. If A contains M, then $A = M \times C$, where C is the centralizer of M (Theorem 1.17). The algebra M has an inner automorphism mapping e_{11} on e_{ii}, and this can be extended, so as to be the identity on C, to all of A. This proves the necessity of our condition.

Conversely, suppose each idempotent e_i has the form e^σ, where $e = e_1$, σ is a suitable automorphism. Decompose e into primitive pairwise orthogonal idempotents of A (Theorem 2.16), as $e = f_1 + \cdots + f_r$. One then computes that the unity of A can be decomposed into the pairwise orthogonal primitive idempotents $1 = \sum f_j^\sigma$, where the sum is taken over all $j = 1, \cdots, r$ and over all σ needed to give the idempotents e^σ. The structure theorem (Theorem 3.19) then asserts that $A = M \times D$, where M is a total matric algebra with basis e_{ij}, and where the diagonal elements e_{kk} are the given idempotents f_j^σ. Each of the original idempotents e_i is the sum of exactly r idempotents f_j^σ. Hence in M we select a subalgebra consisting of those matrices which are constructed from blocks of $r \times r$ scalar matrices. This subalgebra is itself a total matric algebra, and its idempotents are the given e_i.

3. **The product formula.** We now prove the following theorem:

THEOREM 4. *If K is a normal subfield of a normal field N, belonging to the subgroup Δ, then the direct product of two crossed products to N and K is given by*

$$(N, \Gamma, \mathbf{h}) \times (K, \Gamma/\Delta, \mathbf{G}) \sim (N, \Gamma, \mathbf{hg}), \tag{5}$$

where $\mathbf{g}$ is the Δ-symmetric factor set obtained from the given set $\mathbf{G}$ for Γ/Δ by the extension (2).

PROOF. Let C denote the direct product on the left of (5); since C has a unity element, we may regard F as a subfield of C. By the definition of a direct product, C is generated by a subalgebra isomorphic to $(N, \Gamma, \mathbf{h})$, which we can identify with this algebra, and another subalgebra isomorphic to (but not identical with) the second factor $(K, \Gamma/\Delta, \mathbf{G})$; so we may write[4] C as

$$C = (N, \Gamma, \mathbf{h}) \times (K', \Gamma/\Delta, \mathbf{G}'), \tag{6}$$

where the subfields K' and N have only the elements of F in common, where K is equivalent to K' over F under a correspondence $y \leftrightarrow y'$, and where $\mathbf{G}'$ is the map of $\mathbf{G}$ under this isomorphism. Each coset of

[4] In (5), we simply use the ordinary convention, writing the direct product of two algebras not necessarily disjoint; in (6) we have C represented more explicitly as the direct product of two of its subalgebras. The distinction is a familiar one.

Γ/Δ may then be interpreted as an isomorphism of K' under the natural correspondence

$$(y')^\sigma = (y^S)', \qquad y \text{ in } K,\ S \text{ in the coset } \sigma. \tag{7}$$

The crossed product (N, Γ, h) is determined by the formulas (1), while $(K', \Gamma/\Delta, G')$ is determined by similar formulas

$$v_\sigma v_\tau = v_{\sigma\tau} G'_{\sigma,\tau}; \qquad y' v_\sigma = v_\sigma y'^\sigma, \qquad y \text{ in } K.$$

The automorphisms S and σ may be extended to the whole algebra C of (6) by the formulas

$$a^S = u_S^{-1} a u_S, \qquad a^\sigma = v_\sigma^{-1} a v_\sigma, \tag{8}$$

for any a in C. In a direct product, any term in one factor commutes with any term in the other, hence S leaves fixed all elements of K' and σ leaves fixed all elements of N.

The direct product C contains the commutative subalgebra $N \times K'$; since N is separable, this algebra is semi-simple (chap. 3, §7) and as such is the direct sum of fields L_i with idempotents e_i. Since $1 = \sum e_i$, L_i has the form $e_i(N \times K')$. Let L be one of these fields, with idempotent e. Then the mapping $z \rightarrow ez$ carries the elements z of N homomorphically into L; since both are fields, this must be an isomorphism of N to part of L. For similar reasons, $y' \rightarrow ey'$ maps K' isomorphically on part of L. These two mappings agree on the common subfield F of N and K'. Therefore L contains the two fields eK and eK', which are equivalent over eF because K and K' are equivalent over F. Since K is normal over F, this implies that $eK = eK'$. This identity means that for each element y in K there exists an element y^* in K' such that $ey = ey^*$, and such that the mapping $y \rightarrow y^*$ is an equivalence of K to K' over F. Now two mappings $y \rightarrow y'$ and $y \rightarrow y^*$ of K to K' can differ only by an automorphism σ of K' over F, so that we may write $y' = y^{*\sigma}$. One may then compute that the replacement of e by the idempotent e^σ simply replaces y^* by y' in the equation $ey = ey^*$. Furthermore, e^σ is the unity element of the field L^σ, which is a direct summand of $N \times K'$ because σ, as defined by (8), is an automorphism of this algebra. Now change the notation, writing L for L^σ, e for e^σ; we then have in $N \times K'$ a direct summand L with unity e such that

$$ey = ey', \qquad y \text{ in } K, \tag{9}$$

where $y \rightarrow y'$ is the given equivalence of K to K'. Furthermore, $L = eN$, though we do not need this fact.

The idempotents e^σ, for σ in Γ/Δ, are all distinct. For suppose this were not the case; then $e^\sigma = e$ for some $\sigma \neq 1$, so that $e^\sigma y = ey = ey'$

$=e^\sigma y'$. On the other hand, one computes by (9) that

$$e^\sigma y = v_\sigma^{-1} e v_\sigma y = v_\sigma^{-1}(ey)v_\sigma = v_\sigma^{-1}(ey')v_\sigma = e^\sigma y'^\sigma, \tag{10}$$

so that $e^\sigma y' = e^\sigma y'^\sigma$. Since the correspondence $y' \rightarrow e^\sigma y'$ is one-one, this gives $y' = y'^\sigma$ for all y', which means that σ is the identity, contrary to assumption. The distinct idempotents e^σ belong to k distinct summands L^σ of $N \times K'$, where k is the degree of K over F. Since each summand L^σ has at least the degree of N, these summands include all the direct summands of $N \times K'$. Hence every primitive idempotent in $N \times K'$ is one of the idempotents e^σ.

If S is in the coset σ, then $(e^S)^\sigma = e$; for one may compute the effect of multiplying $(e^S)^\sigma$ by an element x^S of K, getting

$$e^{S\sigma} x^S = e^{S\sigma} x^{S\sigma} = (ex)^{S\sigma} = (ex')^{S\sigma} = e^{S\sigma} x'^{S\sigma} = e^{S\sigma} x'^\sigma = e^{S\sigma}(x^S)',$$

where the last transformation uses the definition (7) of the automorphism σ. Since any element y of K can be written in the form $y = x^S$, this proves that $e^{S\sigma} y = e^{S\sigma} y'$, for every y. On the other hand, $e^{S\sigma}$ is a primitive idempotent, hence is e^τ for some τ in Γ/Δ. As in (10), one then computes that $e^{S\sigma} y = e^\tau y = e^\tau y'^\tau$. Compared with the previous equation, this means that $y'^\tau = y'$, hence that $\tau = 1$, hence that $e^{S\sigma} = e$, as asserted.

The conclusion $e^{S\sigma} = e$ may be reinterpreted in terms of the definitions (8) of the extended automorphisms S and σ. It then becomes the assertion that e commutes with the product $u_S v_\sigma$. If S is in the coset σ, we write w_S for $u_S v_\sigma$, and have

$$ew_S = w_S e, \qquad w_S = u_S v_\sigma. \tag{11}$$

The idempotents $e, e^\sigma, e^\tau, \cdots$ of $N \times K'$ are all conjugate in the given algebra C of (6); hence Theorem 3 provides a total matric subalgebra M of C of degree k and with basis e_{ij}, where $e_{11} = e$, $e_{22} = e^\sigma, \cdots$. This algebra is a direct factor of C (Theorem 1.17); so

$$C = (N, \Gamma, \boldsymbol{h}) \times (K', \Gamma/\Delta, \boldsymbol{G}) = M \times B, \tag{12}$$

where B is the C-centralizer of M. By the structure of a total matric algebra eCe will be a subalgebra equivalent to B. This subalgebra contains a subfield $eNe = eN$ isomorphic to N, with automorphisms $ez \leftrightarrow ez^S$, and also contains elements $ew_S e = ew_S$ of (11), one for each automorphism. The multiplication table for these elements may be computed, using the fact that $e\boldsymbol{G} = e\boldsymbol{G}'$; it is

$$(ez)(ew_S) = (ew_S)(ez^S), \qquad (ew_S)(ew_T) = (ew_{ST})(eh_{S,T}G_{\sigma,\tau}),$$

where S and T lie respectively in the cosets σ and τ. Since the whole algebra eCe has the same degree as N, this means simply that eCe is a crossed product to eN and the factor set $eh_{S,T}G_{\sigma,\tau}$ (Albert, p. 67). Therefore (12) proves that C is similar to a crossed product B of the desired form.

4. **Properties of symmetric factor sets.** We now return to the proof of Theorem 2 of the introduction. Given a factor set $\boldsymbol{G}$ of elements in K, one proves at once that the definition (2) for $\boldsymbol{g}$ does yield a factor set for Γ. Conversely, if a given factor set $\boldsymbol{g}$ is Δ-symmetric, the associativity conditions $g_{S,TR}g_{T,R}=g_{ST,R}(g_{S,T})^R$ for R in Δ become

$$g_{S,T}g_{T,1} = g_{ST,1}(g_{S,T})^R.$$

But the associativity conditions with $R=1$ make $g_{T,1}=g_{ST,1}$, so the result above becomes $(g_{S,T})^R=g_{S,T}$; hence each element $g_{S,T}$ of the factor set lies in the subfield K. One may then define $G_{\sigma,\tau}$ by (2), and show that $\boldsymbol{G}$ is a factor set for K. For $\boldsymbol{G}$ so defined, the formula of Theorem 4 gives

$$(N, \Gamma, \boldsymbol{g}^{-1}) \times (K, \Gamma/\Delta, \boldsymbol{G}) \sim (N, \Gamma, \boldsymbol{g}^{-1}\boldsymbol{g}) \sim (N, \Gamma, 1) \sim 1.$$

Multiplying by $(N, \Gamma, \boldsymbol{g})$, one concludes that $(K, \Gamma/\Delta, \boldsymbol{G}) \sim (N, \Gamma, \boldsymbol{g})$, as in the formula (3) of Theorem 2.

Theorem 1 now follows formally from Theorem 2. For, if an algebra $(N, \Gamma, \boldsymbol{h})$ is split by the normal subfield K, it is similar[5] to a crossed product $(K, \Gamma/\Delta, \boldsymbol{G})$, and by Theorem 2 the latter algebra is in turn similar to $(N, \Gamma, \boldsymbol{g})$, where $\boldsymbol{g}$ is the Δ-symmetric extension of $\boldsymbol{G}$. But $(N, \Gamma, \boldsymbol{h}) \sim (N, \Gamma, \boldsymbol{g})$ gives $\boldsymbol{h} \sim \boldsymbol{g}$ (Theorem 5.5), so $\boldsymbol{h}$ is the associate of a Δ-symmetric factor set $\boldsymbol{g}$, as asserted. Conversely, if $\boldsymbol{h}$ is associate to a Δ-symmetric factor set $\boldsymbol{g}$, then

$$(N, \Gamma, \boldsymbol{h}) \sim (N, \Gamma, \boldsymbol{g}) \sim (K, \Gamma/\Delta, \boldsymbol{G}),$$

and the latter algebra is indeed split by K. This completes the proof of Theorem 1.

5. **Arbitrary direct products.** Now we consider two crossed products to any two given fields K and K' which are finite, separable, and normal over a common base field F. The composite $N=K\cup K'$ of K and K' is uniquely determined; we may regard K and K' as subfields of N. The Galois group Γ of N/F is determined in terms of the groups Σ and Σ' of K/F and K'/F as follows.[6]

[5] Because any normal simple algebra A split by a field K normal over F is similar to a crossed product to this field K. This well known fact is contained in the proof of Theorem 5.1.

[6] This result is known, although explicit citations are rare. To prove it, observe that any S induces and is determined by σ and σ', and then count the number of al-

LEMMA. *Let a field N be the join of two subfields K and K', each finite, separable, and normal over a common base field F, so that N is also finite, separable, and normal over F. If σ, σ' are automorphisms of K, K', respectively, which have the same effect on each element of the intersection $K \cap K'$, then there exists one and only one automorphism S of N/F which induces the given automorphisms σ and σ'. Every automorphism S of N/F may be obtained in this way, and the correspondence $S \leftrightarrow (\sigma, \sigma')$ maps the Galois group of N/F isomorphically on a subgroup of the direct product of the Galois groups of K/F and K'/F.*

Our most inclusive result on direct products now is the following theorem:

THEOREM 5. *Let (K, Σ, G) and (K', Σ', G') be any two given crossed products to fields K and K' normal over F. Let $S \leftrightarrow (\sigma, \sigma')$ and $T \leftrightarrow (\tau, \tau')$ be any two automorphisms of the composite field, determined, as in the lemma, in terms of automorphisms of K and K', and extend the given factor sets G and G' to factor sets for $K \cup K'$ by the formulas*

$$g_{S,T} = G_{\sigma,\tau}, \quad g'_{S,T} = G'_{\sigma',\tau'}, \quad S \to (\sigma, \sigma'), \quad T \to (\tau, \tau'). \tag{13}$$

Then the direct product of the two given crossed products is

$$(K, \Sigma, G) \times (K', \Sigma', G') \sim (K \cup K', \Gamma, gg'). \tag{14}$$

In the special case when K and K' are disjoint, the Galois group of $K \cup K'$ is just the direct product of the two groups Σ and Σ' and the formulas (13) mean simply that the matrix of gg' is the Kronecker product of the matrices G and G'. This case has already been considered by one of us.[7] In the case when K is a subfield of K', the formula (14) specializes to the formula derived in Theorem 4. This special case gives a proof of (14) in general, for observe that $(\sigma, \sigma') \to \sigma$ maps Γ homomorphically on Σ, so that the formula (13) really extends G to be a factor set for Γ which is symmetric relative to a suitable subgroup Δ. Therefore $(K, \Sigma, G) \sim (K \cup K', \Gamma, g)$ by (3). The analogous result for K' then gives (14).

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lowable pairs (σ, σ'). If K, K' and $K \cap K'$ have over F the respective degrees k, k', and d, then the degree of $K \cup K'$ over F is kk'/d, and the number of pairs (σ, σ') which agree on the intersection $K \cap K'$ is also kk'/d. Hence every (σ, σ') is realized as an automorphism S of $K \cup K'$.

[7] O. F. G. Schilling, *The structure of certain rational infinite algebras*, Duke Mathematical Journal, vol. 3 (1937), p. 305.