HEAT CONDUCTION IN AN INFINITE COMPOSITE SOLID[1]

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1. **Introduction.** The problem of one-dimensional heat conduction in finite or semi-infinite composite solids has been investigated extensively. The doubly-infinite case, however, seems to have been treated only for special initial temperature distribution functions.[2]

The purpose of the present paper is to treat the general case. The Laplace transformation is used formally in §2 to discover the solution, which is then rigorously established in §3. In §4, finally, a uniqueness theorem is proved, under more restrictive conditions on the initial distribution function.

Lebesgue integrals are used throughout.

2. **The formal solution.** Consider two plane-boundary semi-infinite homogeneous solids, composed of different materials, placed in perfect thermal contact. If the conduction of heat takes place in only one dimension, perpendicular to the interface, the temperature $U(x, t)$ satisfies the following differential system:

$$\frac{\partial U}{\partial t} = a_\nu \frac{\partial^2 U}{\partial x^2}; \qquad t > 0;\ x < 0,\ \nu = 1;\ x > 0,\ \nu = 2; \tag{1}$$

$$\lim_{t \to 0} U(x, t) = f(x); \qquad x \neq 0; \tag{2}$$

$$\lim_{x \to -0} U(x, t) = \lim_{x \to +0} U(x, t); \qquad t > 0; \tag{3}$$

$$\lim_{x \to -0} k_1 \frac{\partial U}{\partial x} = \lim_{x \to +0} k_2 \frac{\partial U}{\partial x}; \qquad t > 0; \tag{4}$$

where x is the perpendicular distance from the interface, t is time, a_ν and k_ν are the thermal diffusivities and conductivities, respectively, of the two materials and are positive constants, and $f(x)$ is a known function, defined for all real x except $x = 0$, whose properties will be specified later.

Denoting the common limit in equation (3) by $\Phi(t)$, the solution can be written in the well known form[3]

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[1] Presented to the Society, April 5, 1941.

[2] Cf. Riemann-Weber, *Die partiellen Differential-Gleichungen der mathematischen Physik*, 5th edition, 1912, vol. 2, p. 98.

[3] Cf. H. S. Carslaw, *The Mathematical Theory of the Conduction of Heat in Solids*, 2d edition, London, 1921, §§18, 23.

(5)[4] $$U(x, t) = V_\nu(x, t) + \frac{|x|}{2(\pi a_\nu)^{1/2}} \int_0^t \Phi(t - \xi)\xi^{-3/2} \exp\left[- \frac{x^2}{4a_\nu \xi} \right] d\xi,$$

where

$$V_1(x, t) = \frac{1}{2(\pi a_1 t)^{1/2}} \int_{-\infty}^0 f(\xi) \left\{ \exp\left[- \frac{(x-\xi)^2}{4a_1 t} \right] - \exp\left[- \frac{(x+\xi)^2}{4a_1 t} \right] \right\} d\xi,$$

and $V_2(x, t)$ is obtained from $V_1(x, t)$ on replacing a_1 by a_2 and integrating from 0 to $+\infty$.

Now apply the Laplace transformation to equations (4) and (5), denoting the transforms of U, V, and so on, by u, v, and so on, respectively:

(4′) $$\lim_{x\to -0} k_1 \frac{\partial u(x, s)}{\partial x} = \lim_{x \to +0} k_2 \frac{\partial u(x, s)}{\partial x},$$

(5′)[5] $$u(x, s) = \phi(s) \exp\left[- |x| (s/a_\nu)^{1/2}\right] + v_\nu(x, s).$$

The unknown function $\phi(s)$ can be eliminated between equations (4′) and (5′), and the solution is then obtained by applying the inverse Laplace transformation to $u(x, s)$:

(6)
$$U(x, t) = V_1(x, t) + \frac{k_1}{A a_1(\pi t)^{1/2}} \int_{-\infty}^0 f(\xi) \exp\left[- \frac{(x+\xi)^2}{4a_1 t}\right] d\xi + \frac{k_2}{A a_2(\pi t)^{1/2}} \int_0^\infty f(\xi) \exp\left[- \frac{(x a_2^{1/2} - \xi a_1^{1/2})^2}{4a_1 a_2 t}\right] d\xi; \quad x < 0, t > 0;$$

$$U(x, t) = V_2(x, t) + \frac{k_2}{A a_2(\pi t)^{1/2}} \int_0^\infty f(\xi) \exp\left[- \frac{(x+\xi)^2}{4a_2 t}\right] d\xi + \frac{k_1}{A a_1(\pi t)^{1/2}} \int_{-\infty}^0 f(\xi) \exp\left[- \frac{(x a_1^{1/2} - \xi a_2^{1/2})^2}{4a_1 a_2 t}\right] d\xi; \quad x > 0, t > 0,$$

where $A = (k_1(a_2)^{1/2} + k_2(a_1)^{1/2})/(a_1 a_2)^{1/2}$.

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[4] Throughout, $x<0$ when $\nu=1$, $x>0$ when $\nu=2$.

[5] The first term in the right member is obtained by the "Faltung" rule. For it and the specific transformations used, cf. G. Doetsch, *Theorie und Anwendung der Laplace Transformation*, Berlin, 1937, particularly the table of transformations in Appendix 2.

3. **The solution established.** We have the following theorem.

THEOREM 1. *If $f(x)$ is Lebesgue integrable over any finite interval, and if there exist two positive numbers B, b, such that $|f(x)| < B \exp |bx|$, then the function $U(x, t)$ defined by equations* (6) *is a solution of the boundary value problem* (1), (3), (4). *The initial condition* (2) *is satisfied uniformly in x in any finite interval $0 < x_1 \leqq |x| \leqq x_2$ in which $f(x)$ is continuous.*

In the classical proofs of similar theorems[6] $f(x)$ is always a bounded function. In adapting such methods of proof to the present situation it is only necessary to investigate the effect of the unboundedness of $f(x)$. A complete proof is given here for only one typical integral in each case. The right members of (6) obviously converge.

(a) *The differential equation.* If $U(x, t)$ is differentiated with respect to t under the integral sign, one of the resulting integrals is of the form

$$J(x, t) \equiv t^{-5/2} \int_0^\infty (x \pm \xi)^2 f(\xi) \exp \left[- \frac{(x \pm \xi)^2}{4t} \right] d\xi.$$

We prove that, if $x > 0$, $0 < t_1 \leqq t \leqq t_2$, $J(x, t)$ converges uniformly in t. From the hypotheses on $f(x)$,

$$| J(x, t) | \leqq B t_1^{-5/2} \int_0^\infty (x \pm \xi)^2 e^{b\xi} \exp \left[- \frac{(x \pm \xi)^2}{4t_2} \right] d\xi.$$

Since the right member converges, and does not contain t, $J(x, t)$ converges uniformly in t.

(b) *The initial condition.* The solutions (6) are composed of two essentially distinct types of integrals, according as x and ξ have the same or opposite signs. We first prove that, in the first case, the integral vanishes with t.

Consider

$$J(x, t) \equiv \frac{1}{2(\pi t)^{1/2}} \int_0^\infty f(\xi) \exp \left[- \frac{(x + \xi)^2}{4t} \right] d\xi, \qquad x > 0.$$

Make the change of variable $\xi = -x + 2\zeta t^{1/2}$, then:

$$J(x, t) = \pi^{-1/2} \int_{x/2t^{1/2}}^\infty f(-x + 2\zeta t^{1/2}) e^{-\zeta^2} d\zeta.$$

From the hypotheses on $f(x)$,

[6] Cf. Carslaw, loc. cit., p. 31; Goursat, *Cours d'Analyse Mathématique*, 4th edition, Paris, 1927, vol. 3, chap. 29.

$$|J(x, t)| \leqq Be^{-bx}\pi^{-1/2}\int_{x/2t^{1/2}}^{\infty} \exp\left[-\zeta(\zeta - 2bt^{1/2})\right]d\zeta.$$

If $x \geqq x_1 > 0$, and $t < x_1/8b$, then

$$|J(x, t)| \leqq \frac{4B}{x_1}(t/\pi)^{1/2} \exp\left[-bx_1 - x_1^2/8t\right].$$

Hence $J(x, t)$ approaches zero with t, uniformly in x in any semi-infinite interval $0 < x_1 \leqq x$.

An inspection of (6) now shows that, in order to prove that the initial condition (2) is satisfied it is sufficient to prove that

$$\lim_{t \to 0} \left[J_1(x, t) - f(x)\right] = 0$$

where

$$J_1(x, t) \equiv \frac{1}{2(\pi t)^{1/2}}\int_0^{\infty} f(\xi) \exp\left[-\frac{(x-\xi)^2}{4t}\right]d\xi, \qquad x > 0,$$

the approach to the limit being uniform in x in any interval $0 < x_1 \leqq x \leqq x_2$. Given any $\epsilon > 0$, we can choose a $\delta > 0$, independent of x and t, such that $|f(\xi) - f(x)| < \epsilon$ if $|x - \xi| < \delta$, $\delta < x_1$. Having chosen δ, we can write

$$(7) \qquad |J_1(x, t) - f(x)| \leqq \int_0^{x-\delta} + \int_{x-\delta}^{x+\delta} + \int_{x+\delta}^{\infty} + \frac{|f(x)|}{\pi^{1/2}}\int_{x/2t^{1/2}}^{\infty} e^{-\xi^2}d\xi,$$

where in each of the first three integrals the integrand is

$$\frac{1}{2(\pi t)^{1/2}}|f(\xi) - f(x)| \exp\left[-\frac{(x-\xi)^2}{4t}\right]d\xi.$$

Denote the right member of (7) by $I_1 + I_2 + I_3 + I_4$. The unboundedness of $f(x)$ is important in I_1, I_3, and I_4. I_2 is easily seen to be less than ϵ. In I_1 make the change of variable $\xi = x - 2\zeta t^{1/2}$. Then

$$I_1 + I_4 \leqq 3Be^{bx_2}\pi^{-1/2}\int_{\delta/2t^{1/2}}^{\infty} e^{-\zeta^2}d\zeta,$$

which obviously approaches zero with t, uniformly in x.

From the hypotheses on $f(x)$,

$$I_3 \leqq \frac{B}{(\pi t)^{1/2}}\int_{x+\delta}^{\infty} \exp\left[b\xi - \frac{(x-\xi)^2}{4t}\right]d\xi.$$

Make the change of variable $\xi = x + 8t\zeta/\delta$; if $t < \delta/8b$,

$$I_3 \leqq \frac{8B}{\delta} e^{bx_2}(t/\pi)^{1/2} \int_{\delta^2/8t}^{\infty} e^{-\zeta} d\zeta,$$

which obviously approaches zero with t, uniformly in x.

(c) *The interface conditions.* Equations (3) and (4) will obviously be satisfied if x can approach zero within the integral signs. The proof is given for one typical case:

$$J(x, t) \equiv t^{-3/2} \int_0^{\infty} (x \pm \xi) f(\xi) \exp \left[- \frac{(x \pm \xi)^2}{4t} \right] d\xi.$$

If $0 < t_1 \leqq t \leqq t_2$, $x > 0$,

$$\begin{aligned} | J(x, t) - J(0, t) | \leqq\ & x t^{-3/2} \int_0^{\infty} | f(\xi) | \exp \left[- \frac{(x - \xi)^2}{4t} \right] d\xi \\ & + t^{-3/2} \int_0^{\infty} \xi | f(\xi) | \exp \left[\frac{-\xi^2}{4t} \right] \left| 1 - \exp \left[- \frac{x^2 - 2x\xi}{4t} \right] \right| d\xi. \end{aligned} \tag{8}$$

Denote the right member of (8) by $I_1 + I_2$. In I_1 make the change of variable $\xi = x + 2\zeta t^{1/2}$:

$$I_1 \leqq \frac{2Bx}{t} e^{bx} \int_{-x/2t^{1/2}}^{\infty} \exp \left[- \zeta(\zeta - 2bt^{1/2}) \right] d\zeta.$$

If the range of integration is split at $\zeta = 1 + 2bt^{1/2}$, it is easily seen that

$$I_1 \leqq \frac{2Bx}{t_1} e^{bx} \left\{ 1 + \pi^{1/2} \exp \left[2b(1 + 2bt_2^{1/2}) t_2^{1/2} \right] \right\},$$

which approaches zero with x, uniformly in t.

Since $|1 - \exp y| \leqq |y| \exp |y|$, we have

$$I_2 \leqq Bxt^{-3/2} \int_0^{\infty} \xi \left| \frac{x - 2\xi}{4t} \right| \exp \left[b\xi - \frac{\xi^2}{4t} + \left| \frac{x^2 - 2x\xi}{4t} \right| \right] d\xi.$$

Split the range of integration at $\xi = 3 + 4bt$; if $x < 1$,

$$\begin{aligned} I_2 \leqq Bxt_1^{-5/2} \exp \left[\frac{x^2}{4t_1} \right] \Bigg\{ & \int_0^{3+4bt_2} \xi(1 + 2\xi) \exp \left[b\xi + \frac{\xi}{2t_1} \right] d\xi \\ & + \int_{3+4bt_1}^{\infty} \xi(1 + 2\xi) \exp \left[- \xi/4t_2 \right] d\xi \Bigg\}, \end{aligned}$$

which approaches zero with x, uniformly in t.

4. **Uniqueness.** We first state the following definition.

DEFINITION. *A function $U(x, t)$ defined for all real x and positive t will be said to satisfy conditions* A *if*:

(A_1) *$U(x, t)$ is a continuous function of x in $-\infty < x < \infty$ for any fixed $t > 0$.*

(A_2) *$U(x, t)$ is a continuous function of t in any finite interval $0 < t_1 \leqq t \leqq t_2$, the continuity being uniform with respect to x in any finite interval $x_1 \leqq x \leqq x_2$.*

(A_3) *$U(x, t)$ is bounded for all real x, all positive t.*

(A_4) *$\partial U(x, t)/\partial t$ exists and satisfies* (A_1) *and* (A_2).

(A_5) *$\partial U(x, t)/\partial x$ and $\partial^2 U(x, t)/\partial x^2$ exist and are continuous functions of x in $-\infty < x < 0$ and $0 < x < \infty$, for any fixed $t > 0$.*

(A_6) *$|\partial U(x, t)/\partial x| < M/t^{1/2}$, $t > 0$, $x \neq 0$; M is a constant independent of t and x.*

(A_7) *$U(x, t) \to 0$ as $|x| \to \infty$, uniformly in t in any finite interval $0 < t \leqq t_1$.*

(A_8) *$U(x, t) \to 0$ as $t \to 0$, uniformly in x in any finite interval $0 < x_1 \leqq |x| \leqq x_2$.*

THEOREM 2. *The function $U(x, t) \equiv 0$ is the only function satisfying conditions* A *that is a solution of the boundary value problem* (1)–(4), *with $f(x) \equiv 0$.*

PROOF. Consider

$$J(x, t) \equiv \frac{1}{2} \int_{-x}^{x} \frac{k_\nu}{a_\nu} [U(\xi, t)]^2 d\xi, \qquad x > 0, t > 0.$$

Assume that, for some $x_1 \neq 0$, $t_1 \neq 0$, $|U(x_1, t_1)| = C > 0$. From condition (A_1), $|U(x, t_1)| > C/2$ in some η-neighborhood of x_1. Hence $J(x, t_1) > D$, any $x > |x_1| + \eta$, $t_1 > 0$, where

$$D = \frac{k_\nu C^2 \eta}{4 a_\nu}; \qquad \eta < |x_1|; \ \nu = 1 \text{ if } x_1 < 0; \ \nu = 2 \text{ if } x_1 > 0.$$

If M is the bound of condition (A_6), we can choose a fixed $x_2 > |x_1| + \eta$ such that

$$(9) \qquad \begin{aligned} |U(x_2, t)| &< \frac{D}{4 k_2 M t_1^{1/2}}, & 0 < t \leqq t_1, \\ |U(-x_2, t)| &< \frac{D}{4 k_1 M t_1^{1/2}}, & 0 < t \leqq t_1. \end{aligned}$$

Having chosen x_2, consider $J(x_2, t)$. From conditions (A_1), (A_2), (A_4),

$J(x_2, t)$ can be differentiated with respect to t under the integral sign; using equation (1),

$$\frac{\partial J(x_2, t)}{\partial t} = \int_{-x_2}^{x_2} k_\nu U(x, t) \frac{\partial^2 U(x, t)}{\partial x^2} dx.$$

From condition (A_5) this may be integrated by parts; using equations (3) and (4),

$$\frac{\partial J(x_2, t)}{\partial t} = \left[k_\nu U(x, t) \frac{\partial U(x, t)}{\partial x} \right]_{x=-x_2}^{x=x_2} - \int_{-x_2}^{x_2} k_\nu \left[\frac{\partial U(x, t)}{\partial x} \right]^2 dx,$$
$$0 < t \leqq t_1.$$

Since the last term is nonpositive, $\partial J/\partial t$ is not greater than the sum of the first two terms. Hence, from (A_6) and equations (9)

$$\frac{\partial J(x_2, t)}{\partial t} < \frac{D}{2(t_1 t)^{1/2}}, \qquad 0 < t \leqq t_1.$$

From conditions (A_3), (A_8), $J(x_2, +0) = 0$. Therefore

$$J(x_2, t_1) < \frac{D}{2t_1^{1/2}} \int_0^{t_1} t^{-1/2} dt = D.$$

But $J(x_2, t_1) > D$. Thus the assumption that $U(x_1, t_1) \neq 0$ leads to a contradiction if $x_1 \neq 0$, $t_1 \neq 0$, and by the continuity conditions (A_1) and (A_8) the theorem is proved.

We are now in a position to prove a more general uniqueness theorem.

DEFINITION. *Let $f(x)$ be defined and continuous for all real x except $x = 0$. A function $U(x, t)$ will be said to satisfy conditions* B *with respect to $f(x)$ if*:

(B_1) *$U(x, t)$ satisfies* (A_1)–(A_6) *inclusive.*

(B_2) *$U(x, t) \to f(x)$ as $t \to 0$, uniformly in x in any finite interval $0 < x_1 \leqq |x| \leqq x_2$.*

(B_3) *$U(x, t) - f(x) \to 0$ as $|x| \to \infty$, uniformly in t in any finite interval $0 < t \leqq t_1$.*

THEOREM 3. *Let $f(x)$ be defined and continuous for all real x except $x = 0$. Then the boundary value problem* (1)–(4) *has at most one solution satisfying conditions* B *with respect to $f(x)$.*

PROOF. If there are two solutions, let $U(x, t)$ be their difference. Then $U(x, t)$ satisfies conditions A and hence vanishes identically.

Finally, it remains to determine conditions on $f(x)$ under which the solution $U(x, t)$ defined by equations (6) will satisfy conditions B with respect to $f(x)$. Sufficient conditions are given by the following theorem.

THEOREM 4. *Let $f(x)$ be continuous except at $x=0$, and bounded for all real x, and let the following limits exist*:

$$f_1 = \lim_{x\to-\infty} f(x), \qquad f_2 = \lim_{x\to+\infty} f(x).$$

Then $U(x, t)$ as defined by equations (6) *satisfies conditions* B *with respect to $f(x)$.*

PROOF. Conditions (A_1)–(A_5) and (B_2) are easily seen to be satisfied. We give the proof for (A_6) and (B_3). An inspection of (6) shows that it is sufficient to prove (A_6) for integrals of the form

$$J(x, t) \equiv \frac{1}{2(\pi t)^{1/2}}\int_0^\infty f(\xi)\exp\left[-\frac{(x\pm\xi)^2}{4t}\right]d\xi, \qquad x>0,\ t>0.$$

By differentiating inside the integral sign and splitting the range of integration at $x=\xi$ it is easily found that

$$\left|\frac{\partial J(x, t)}{\partial x}\right| < \frac{N}{(\pi t)^{1/2}},$$

where N is the upper bound of $f(x)$.

For (B_3) it is sufficient to prove that

$$J_1(x, t) \to 0 \qquad \text{as } x\to+\infty, \tag{10}$$

$$J_2(x, t) - f(x) \to 0 \qquad \text{as } x\to+\infty, \tag{11}$$

the approach being uniform in t in $0<t\leqq t_1$, where

$$J_1(x, t) \equiv t^{-1/2}\int_0^\infty f(\xi)\exp\left[-\frac{(x+\xi)^2}{4t}\right]d\xi, \qquad x>0,\ t>0,$$

$$J_2(x, t) \equiv \frac{1}{2(\pi t)^{1/2}}\int_0^\infty f(\xi)\exp\left[-\frac{(x-\xi)^2}{4t}\right]d\xi, \qquad x>0,\ t>0.$$

To prove (10) make the substitution $\xi=-x+2\zeta t^{1/2}$ in J_1:

$$|J_1(x, t)| \leqq 2N\int_{x/2t^{1/2}}^\infty e^{-\zeta^2}d\zeta < \frac{4Nt_1^{1/2}}{x}\exp\left[\frac{-x^2}{4t_1}\right],$$

which clearly approaches zero as x approaches infinity, uniformly in t in $0<t\leqq t_1$.

To prove (11), we have

$$J_2(x, t) - f(x) = \frac{1}{2(\pi t)^{1/2}} \int_0^\infty \{f(\xi) - f(x)\} \exp\left[-\frac{(x-\xi)^2}{4t}\right] d\xi + \frac{f(x)}{\pi^{1/2}} \int_{x/2t^{1/2}}^\infty e^{-\xi^2} d\xi, \qquad x > 0,\ t < t \leqq t_1.$$

In the first term of the right member, make the change of variable $\xi = x + 2\zeta t^{1/2}$. Then

$$\begin{aligned} |J_2(x, t) - f(x)| \leqq{} & \frac{3N}{\pi^{1/2}} \int_{x/4t^{1/2}}^\infty e^{-\zeta^2} d\zeta \\ & + \frac{1}{\pi^{1/2}} \int_{-x/4t^{1/2}}^\infty |f(x) - f(x + 2\zeta t^{1/2})| e^{-\zeta^2} d\zeta. \end{aligned} \tag{12}$$

Since $f(x)$ approaches a limit as $x \to \infty$, the second term of the right member of (12) can be made less than any $\epsilon > 0$ by choosing $x > X$, independent of t. The first term of the right member of (12) clearly approaches zero as required in order that (11) be true.

It may be remarked that the conditions imposed on $f(x)$ in Theorem 4 are merely sufficient, and not necessary. In particular, $f(x)$ need not have limits as $|x| \to \infty$. For example, $U(x, t) - x$ as defined by (6) satisfies conditions B if $f(x) = x$. However, in most physical applications $f(x)$ will satisfy the hypotheses of this theorem, since the problem considered here is an approximation, for small time values, to the finite composite solid problem.

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