

arithmetical relations concerning the Bernoulli and allied numbers. It depends mainly on the following idea: Let a and b be rational, with $a \equiv b \pmod{p}$, where p is any prime integer. If a and b are independent of p it then follows, since there is an infinity of primes, that $a = b$. This is applied in connection with what is perhaps the simplest formula in which a single Bernoulli number appears: $S_n^c \equiv pb_n \pmod{p^2}$, where $S_n(p) = 1^n + 2^n + \cdots + (p-1)^n$. The paper starts with simple identities involving $(x^p - 1)/(x - 1)$, obtains congruences modulo p or p^2 by differentiation and integration, and then makes substitutions for the variables which give congruences involving the Bernoulli numbers. Generalizations of the Staudt-Clausen theorem as well as analogues of the same are obtained. Proofs are given of nearly all the results which were stated without proof in two previous papers by the writer (Proceedings of the National Academy of Sciences, vol. 23 (1937), pp. 555-559; vol. 25 (1939), pp. 197-201). This article will appear in the Transactions of this Society. (Received August 12, 1941.)

ANALYSIS

467. Lipman Bers: *A convergence theorem for analytic functions of two variables.*

Let M be a domain of the four-dimensional z_1, z_2 -space ($z_k = x_k + iy_k, k = 1, 2$) bounded by the two analytic hypersurfaces $E[z_1 \in B(z_2) + c(z_2), z_2 = e^{i\theta}, 0 \leq \theta < 2\pi]$ and $E[z_1 \in c(z_2), |z_2| < 1]$ where $B(z_2)$ is a star domain bounded by the curve $c(z_2) = E[z_1 = h(z_2, \lambda), 0 \leq \lambda < 2\pi]$ and $h(z_2, \lambda)$ is an analytic function of $z_2, |z_2| \leq 1$, for every fixed value of λ , and is subject to certain additional conditions. Consider a sequence of analytic functions $\{f_n(z_1, z_2)\}$ defined in M and satisfying the condition $\int_0^{2\pi} \int_0^{2\pi} |f[r_1 h(r_2 e^{i\theta}, \lambda), r_2 e^{i\theta}]|^p d\theta d\lambda < K, 0 < r_k < 1, k = 1, 2, n = 1, 2, \dots$, where p is a fixed number greater than 1. It is known that $f_n(z_1, z_2)$ possesses finite sectorial limits $(F_n(\theta, \lambda))$ almost everywhere on the intersection $F = E[z_1 = h(e^{i\theta}, \lambda), z_2 = e^{i\theta}, 0 \leq \theta < 2\pi, 0 \leq \lambda \leq 2\pi]$ of the two boundary hypersurfaces of M (see Bergman and Marcinkiewicz, *Fundamenta Mathematicae*, vol. 33 (1939), pp. 75-94, and a paper by the author to appear in the *American Journal of Mathematics*). If the sequence $\{F_n(\theta, \lambda)\}$ converges in a set of positive two-dimensional measure, the sequence $\{f_n(z_1, z_2)\}$ converges uniformly in every closed subdomain of M . This theorem can be generalized, in a modified form, for more general types of domains. (Received September 27, 1941.)

468. A. B. Brown: *Independent parameters for sets of functions.*

The results previously announced for the case of one function of n variables and m parameters (under the title *On the number of independent parameters*, abstract 46-11-485) are now extended to the case of a set of r functions of n variables and m parameters. The methods and results are similar to those for the case of one function. An additional theorem is given, on invariance of the number of parameters in a "complete" set under certain transformations of parameters. Only one paper is offered for publication, covering the more general case announced here, but under the original title. (Received September 11, 1941.)

469. E. R. Lorch: *The theory of analytic functions in normed abelian vector rings.* Preliminary report.

A complex vector space \mathfrak{R} for which the commutative multiplication of elements is defined is here called a normed abelian vector ring if the norm satisfies

$|A \cdot B| \leq |A| \cdot |B|$. \mathcal{R} is assumed to possess a unit I for which $|I| = 1$. For those rings in which the singular elements (those without inverse) are precisely the frontier of the group of nonsingular elements, the classic theory of analytic functions is developed. Analytic functions are defined to be those mapping domains in \mathcal{R} upon \mathcal{R} for which the derivative exists. The theory elaborated for these functions includes the Cauchy theorem and formula and the power series representation. In addition, various known results on normed rings are derived anew by the author's methods. (Received September 3, 1941.)

470. A. N. Lowan and William Horenstein: *On the function*
 $H(m, a, x) = \exp(-ix)F(m+1-ia, 2m+2, 2ix)$.

The substitutions $\alpha = m+1-ia$, $\gamma = 2m+2$ and $u = 2ix$, in a known integral representation of the confluent hypergeometric function $F(\alpha, \gamma, x)$ led to the following interesting results: (1) A recurrence formula between the values of $H(m, a, x)$ for three consecutive integral values of the parameter m . (2) A recurrence formula between the values of $H(m, a, x)$ and $(\partial/\partial x)H(m, a, x)$ for two consecutive integral values of parameter m . (3) A recurrence formula between $H(m, a, x)$ and $(\partial/\partial x)H(m, a, x)$ for the integral value of the parameter m , and the value of $H(m+1, a, x)$. (4) A recurrence formula between the value of $H(m, a, x)$ for an integral value of the parameter m and the values of $H(m+1, a, x)$ and $(\partial/\partial x)H(m+1, a, x)$. The above results were obtained in the course of some exploratory study of the function $H(m, a, x)$ now being carried out by the Mathematical Tables Project conducted by the Work Projects Administration for the City of New York under the sponsorship of the National Bureau of Standards. (Received August 5, 1941.)

471. Brockway McMillan: *Absolutely monotone functions*. Preliminary report.

Let \mathcal{A} be an additive complemented family of subsets of some space S . Let \mathcal{A}^N be the additive complemented family determined by the Cartesian product sets $a_1 \otimes \cdots \otimes a_N$. Let \mathcal{P} be the class of finite partitions π of S by pairwise disjoint members of \mathcal{A} , \mathcal{P} being ordered by refinement. If $F(x)$ is defined for $x \in \mathcal{A}$, let $\Delta^0 F(x) = F(x)$ and $\Delta^N F(x; a_1 \otimes \cdots \otimes a_N) = \Delta^{N-1} F(x+a_N; a_1 \otimes \cdots \otimes a_{N-1}) - \Delta^{N-1} F(x; a_1 \otimes \cdots \otimes a_{N-1})$. If these differences are non-negative for every N and every collection of pairwise disjoint $a_i \in \mathcal{A}$ and if $u \in \mathcal{A}^N$, the limit on \mathcal{P} $F^N(u) = \lim \{ \sum \Delta^N F(0; a_1 \otimes \cdots \otimes a_N \cdot u) \mid \pi \in \mathcal{P} \}$ exists for every N , the summation being taken over all combinations $a_1 \otimes \cdots \otimes a_N$ of distinct elements of π . If $G(x) = F^N(x^N)$, then $\Delta^{N+1} G(x; a_1 \otimes \cdots \otimes a_{N+1}) = 0$ identically in x and pairwise disjoint a_i . Furthermore, (i) $F(x) \geq F(0) + \sum (1/n!) F^N(x^N)$. If $\Delta^{K+1} F = 0$ identically, (i) is an equality and the right member terminates with the term $F^K(x^K)$. If (i) is an equality for some $x = x_0$, then equality holds for all $x \leq x_0$. There exist families \mathcal{A} and functions F for which (i) is not identically an equality. (Received August 14, 1941.)

472. G. W. Mackey: *On the lattice of closed linear subspaces of a normed linear space*.

Let X_1 and X_2 be normed linear spaces. Let L_1 and L_2 be the lattices of closed linear subspaces of X_1 and X_2 respectively. If L_1 and L_2 are isomorphic as lattices then, X_1 and X_2 are isomorphic as normed linear spaces, i.e., there exists a 1-1 linear transformation of all of X_1 into all of X_2 which is a homeomorphism. Eidelheit

(Studia Mathematica vol. 9 (1940) pp. 97-105) has proved an analogous theorem in which the ring of continuous linear transformations of the space into itself takes the place of the lattice of closed linear subspaces. He assumes in addition that X_1 and X_2 are complete. A strengthened form of Eidelheit's theorem in which it is not necessary to assume the completeness of X_1 and X_2 follows readily from the two lemmas on which the lattice theorem is based. (Received July 21, 1941.)

473. J. F. Paydon: *Convergence regions and value regions for continued fractions*. Preliminary report.

The author shows that if U is the circular region $|z| \leq r$, $r \leq 1/4$, and V is the circular region $|z - \alpha| \leq \beta$ where $\alpha = [1 + 2r - (1 - 4r)^{1/2}] / (4r + 2r^2)$, $\beta = [1 - r - (1 + r)(1 - 4r)^{1/2}] / (4r + 2r^2)$, and if a_2, a_3, \dots lie in U , then the continued fraction $f = 1/1 + a_2/1 + a_3/1 + \dots$, which is necessarily convergent, has its value in V . Moreover if v is on the boundary of V , then there is one and only one continued fraction with elements in U whose value is v . This uniqueness property is also established for the parabolic element region and companion value region of Scott and Wall (abstract 46-11-503). The following convergence theorem is also proved: If the elements of f lie in the region $R(z) \geq 0$, $I(z) \geq 0$, $|z| - R(z) \leq 1$, then f converges if and only if (a) some a_n is 0, or (b) $a_n \neq 0$, ($n = 2, 3, 4, \dots$), and $\sum |b_n|$ diverges where, $b_1 = 1$, $a_n = 1/b_{n-1}b_n$, ($n = 2, 3, 4, \dots$). (Received August 7, 1941.)

474. Maxwell Reade and E. F. Beckenbach: *Square averages and a class of harmonic polynomials*.

Harmonic functions are characterized by the property that they are identical with their own circular averages, both areal and peripheral. The authors delineate the classes of functions which are identical with their own: (a) square averages, (b) rectangular averages. The first class is a class of harmonic polynomials, while the latter class is a more general class of polynomials. A typical result is the following: If the function $f(x, y)$ is superficially summable over a finite simply connected domain D , then a necessary and sufficient condition that $f(x, y) = (1/4h^2) \int_{-h}^h \int_{-h}^h f(x + \xi, y + \eta) d\xi d\eta$ hold for each point (x, y) in D , for all h sufficiently small, is that $f(x, y)$ be a harmonic polynomial of the form $f(x, y) = A_0 + A_1x + A_2y + A_3xy + A_4(x^2 - y^2) + A_5(x^3 - 3xy^2) + A_6(3x^2y - y^3) + A_7(x^3y - y^3x)$, where the A_i are constants. (Received August 27, 1941.)

475. J. F. Ritt: *Complete difference ideals*.

This paper appeared in full in the American Journal of Mathematics for October, 1941. (Received September 19, 1941.)

476. Rubin Schatten: *On the direct product of Banach spaces*.

Two Banach spaces E_1 and E_2 may be combined in two ways: the well known "sum" $E_1 \oplus E_2$, and $E_1 \otimes E_2$, where \otimes refers to the distributive operation. The present paper deals with the second construction in its algebraic aspects, presents a theory of possible crossnorms, and considers the conjugate spaces and their crossnorms. The existence of a greatest crossnorm and a least crossnorm whose conjugate is also a crossnorm has been proved, and that they are conjugate to each other. A self-conjugate crossnorm has been constructed which is a generalization of the norm for Hilbert spaces given by F. J. Murray and John von Neumann (Annals of Mathematics, (2), vol. 37 (1936), p. 127). (Received September 26, 1941.)

477. H. S. Wall: *The behavior of certain Stieltjes continued fractions near the singular line.*

The author shows that the function $f(z)$ represented by a continued fraction of the form $1/1+g_1z/1+(1-g_1)g_2z/1+(1-g_2)g_3z/1+\dots$ in which $0 < g_n < 1$, $n=1, 2, 3, \dots$, and the series $\sum |1-2g_n|$ converges, approaches a finite limit $\alpha(s)$ as $z \rightarrow -s$, $s > 1$, along any path in the upper half-plane, and the limit $\bar{\alpha}(s)$, the complex conjugate of $\alpha(s)$, as $z \rightarrow -s$ from the lower half-plane. The function $\alpha(s)$ is continuous and not real, $s > 1$. (Received August 18, 1941.)

APPLIED MATHEMATICS

478. Arthur Korn: *On the problem of pulsating spheres in a liquid.*

The problem of the pulsating spheres treated for the first time almost 80 years ago by the Norwegian mathematician C. A. Bjerknes and chosen by the author as the point of departure for a mechanical theory of gravitation and the electromagnetic field, must be considered as a fundamental problem of mathematical physics. Interactions inversely proportional to the square of the distance can solely be produced by such spheres changing their volumes periodically with approximately the same frequency in an incompressible or approximately incompressible fluid. The sign of the forces of interaction (attraction in the case of equal phases, repulsion in the case of opposite phases) is a certain difficulty for the hypothesis that electrical particles are pulsating spheres. The sign can be changed, if one imposes on the spheres the condition that they maintain the amplitudes of their pulsations at any displacement of the spheres, a condition which is not imposed in Bjerknes' problem. The calculations are a little complicated, if one has to find out the pressure at the surfaces of the spheres and if one derives from these pressures the forces in question. Here is given a rather simple way of deriving the forces from the energy of the liquid in the case of displacements of the spheres, and one can easily understand the change of the sign, if one imposes on the spheres the condition that they maintain the amplitudes of their pulsations at any displacement. (Received September 22, 1941.)

GEOMETRY

479. George Comenetz: *Isotropic curves on a surface.*

Let S be a regular analytic surface in complex three-dimensional euclidean space, and O a point of S at which the tangent plane T is isotropic. Such a point is singular for the differential equation $ds^2=0$ of the isotropic net of curves on S . The problem is to describe the regular analytic isotropic curves on S that pass through O . Let L be the isotropic straight line in T through O , and p its order of contact with S . Draw an arbitrary curve F on S having contact of order p with L at O . Draw the normal to S at a point V of F , and let q be the "order of coincidence" of the normal with its limit L , as V approaches O . Then if $q > p/2$ the number of isotropic curves is 0 or 2, if $q = p/2$ it is 1 or 2 or $1 + \infty^1$, and if $q < p/2$ it is 2. The conditions under which the different possibilities are realized, and the orders of contact of the curves with L and one another, are stated. (Received September 15, 1941.)

480. J. J. DeCicco: *Bi-isothermal systems of curves.*

A generalization of the concept of isothermal family of curves in four-dimensional space S_4 is obtained in this paper. Kasner has termed the correspondences of S_4 , de-