A CHARACTERIZATION OF THE DISC

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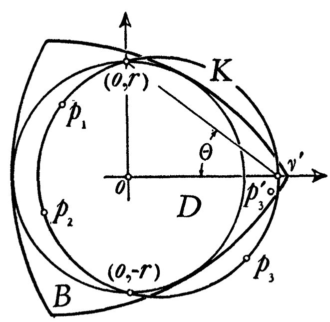
In this paper the disc[2] is characterized as the *only* connected, simply connected domain[3] B with the following property, C_3: B will cover (by an isometry) any subset P of the plane provided B will cover each 3 points of P.

The disc has property C_3. For a plane set P can be covered by a ρ-disc if and only if the members of the family F of ρ-discs with centers in P have a common point. If now each three points of P are



on a ρ-disc then each three discs of F intersect and by a theorem on convex bodies due to E. Helly[4] there is a point common to all the discs of F.

LEMMA. *A bounded, closed subset of the plane contains a largest circle.*

The proof is accomplished by selecting a sequence of circles from the set whose centers converge to a point and whose radii converge to the least upper bound of the radii of circles in the set and (using the fact that the set is closed) showing that the limiting circle belongs to the set.

THEOREM. *The disc is the only connected, simply connected domain with property* C_3.

PROOF. We assume B to be the given domain, and show that B

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[2] The disc of radius ρ and center p is the set of points x of E_2 such that $px \leqq \rho$.

[3] Closure of a bounded open subset of E_2.

[4] *Theorem.* If each $n+1$ sets of a family of *bounded*, closed, convex subsets of E_n intersect, there is a point common to all the sets. Jahrbuch der Deutschen Mathematiker-Vereinigung, vol. 32 (1932), pp. 175–176.

must be a disc. By the preceding lemma, B contains a largest circle (center o and radius r), and since B is simply connected, the whole disc D of the circle is contained in B. If we suppose B is not a disc, there is a point p of B outside D, and, since B is connected, a point v' of B such that $r < ov' \leqq \max(op, 3^{1/2}r)$. (The inequality on the right insures that $\theta \geqq 30°$ in the figure.) Introduce coordinates in the plane with o as origin and the ray ov' as positive X-axis. Then D is defined by the inequality

$$(1) \qquad D: \quad x^2 + y^2 \leqq r^2,$$

and the circle K of the three points $(0, \pm r)$, v' has the equation

$$(2) \qquad K: \quad \left(x - \left(\frac{v^2 - r^2}{2v}\right)\right)^2 + y^2 = \left(\frac{v^2 + r^2}{2v}\right)^2$$

where v is the abscissa of the point v'.

The circle K has the following properties:

(α) The radius of K is greater than r. For, since $v > r$,

$$\left(\frac{v^2 + r^2}{2v}\right) = \left(\frac{v^2 - r^2}{2v}\right)^2 + r^2 > r^2.$$

(β) The arc of K to the left of the Y-axis is of length $\geqq 120°$. For the arc is $2(2\theta) = 4\theta \geqq 120°$ by choice of v'.

(γ) The arc of K to the left of the Y-axis is in D. From (2)

$$(3) \qquad x^2 + y^2 = r^2 + x\left(\frac{v^2 - r^2}{v}\right)$$

so that for $x \leqq 0$ the inequality (1) is satisfied by points of K.

(δ) The maximum distance of a point of K from the origin is v. This follows easily from (3).

If, now, any three points p_1, p_2, p_3 of K are selected, at least two of them, say p_1, p_2, will lie on an arc of length $\leqq 120°$. By property (β), K can then be rotated about its center so that this arc will fall to the left of the Y-axis, that is, $p_1, p_2 \in D$ by property (γ). Since $op_3 \leqq v$ by property (δ), there is, since B is connected, a point p_3' of B at distance op_3 from o. Rotate K about the origin so that p_3 goes into p_3'. Since p_1, p_2 will remain in D, the three points p_1, p_2, p_3 are seen to be congruent to three points of B. But p_1, p_2, p_3 are *any* three points of K, so that B must, by our assumption, cover K. By property (α) this is impossible, and hence no point p of B can lie outside D, that is, B is the disc D.

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