MEAN-VALUE SURFACES

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Introduction. The real functions

$$(1) \qquad x_j = x_j(u, v), \qquad j = 1, 2, 3,$$

defined and continuous in a finite simply connected domain[1] D, will be said to define a surface S. If the first partial derivatives of the functions (1) are continuous in D, and if

$$(2) \qquad E(u, v) = G(u, v), \qquad F(u, v) = 0$$

hold in D, where

$$E(u, v) \equiv \sum_{j=1}^{3} x_{ju}^2, \qquad F(u, v) \equiv \sum_{j=1}^{3} x_{ju}x_{jv}, \qquad G(u, v) \equiv \sum_{j=1}^{3} x_{jv}^2$$

are the coefficients of the first fundamental quadratic form of S, then the surface is said to be given in isothermic representation by the functions (1) and the parameters u, v are said to be isothermic parameters; the map of D on S is conformal except where $E=G=0$.

In a previous paper,[2] the authors studied the equation

$$(3) \qquad \sum_{j=1}^{3}\left[\int_C x_j(u, v)dz\right]^2 = 0, \qquad z = u + iv,$$

where C is a circle in D; the following necessary and sufficient condition was obtained.

THEOREM A. *If the functions* (1) *have continuous partial derivatives of the third order in a finite simply connected domain D, then a necessary and sufficient condition that they map D isothermically either on a surface S that lies on a sphere of finite non-null radius, such that circles are mapped on circles, or on a minimal surface S, is that* (3) *hold for each circle C in D.*

1. **Mean-value surfaces.** Let the coordinate functions (1) of a surface S be continuous in a finite simply connected domain D; then the circular averages

[1] A domain is a non-null connected open set.

[2] *Generalizations to space of the Cauchy and Morera theorems*, Transactions of this Society, vol. 49 (1941), pp. 354–377; in particular, see p. 365.

$$(4)\quad x_j = A_{j,\rho}(u, v) \equiv \frac{1}{\pi\rho^2}\iint_{\xi^2+\eta^2\leqq\rho^2} x_j(u+\xi, v+\eta)d\xi d\eta, \quad j = 1, 2, 3,$$

where ρ is a positive constant, will be said to define *a mean-value surface S_ρ associated with S.* We define

$$A_{j,0}(u, v) \equiv x_j(u, v), \qquad j = 1, 2, 3.$$

We note that the functions (4) are defined and have continuous partial derivatives of the first order in an open set of points D_ρ which is interior to D; since D_ρ is not necessarily a connected set, S_ρ may consist of several pieces.

THEOREM 1.1. *If the functions* (1) *are continuous in a simply connected domain D, then a necessary and sufficient condition that* (3) *hold for each circle C in D is that all mean-value surfaces S_ρ associated with the surface S, defined by the functions* (1), *be given in isothermic representation by* (4).

PROOF. The first partial derivatives of the functions (4) are given by the relations[3]

$$\frac{\partial A_{j,\rho}}{\partial u} = \frac{1}{\pi\rho^2}\int_{\xi^2+\eta^2=\rho^2} x_j(u+\xi, v+\eta)d\eta,$$
$$\frac{\partial A_{j,\rho}}{\partial v} = -\frac{1}{\pi\rho^2}\int_{\xi^2+\eta^2=\rho^2} x_j(u+\xi, v+\eta)d\xi, \quad j = 1, 2, 3,$$

which are valid for points of D_ρ; hence

$$(5)\quad \sum_{j=1}^{3}\left[\int_{\xi^2+\eta^2=\rho^2} x_j(u+\xi, v+\eta)(d\xi + id\eta)\right]^2 = -\pi^2\rho^4[E_\rho - G_\rho + 2iF_\rho],$$

where E_ρ, F_ρ and G_ρ are the coefficients of the first fundamental quadratic form of S_ρ. From (2), (3) and (5) we obtain the theorem.

From Theorems A and 1.1 we obtain the following result.

THEOREM 1.2. *If the functions* (1) *have continuous partial derivatives of the third order in a finite simply connected domain D, then a necessary and sufficient condition that they map D isothermically either on a surface S that lies on a sphere of finite non-null radius, such that circles are mapped on circles, or on a minimal surface S, is that all mean-value surfaces S_ρ associated with the surface S defined by the functions* (1) *be given in isothermic representation by* (4).

[3] T. Radó, *Subharmonic Functions*, Berlin, 1937, p. 11.

2. **Mean-value surfaces and transformations of axes.** In §3 we shall make use of the following observations.

2.1. S_ρ is invariant under rigid transformations in the (x_1, x_2, x_3)-space; if

$$x_j = a_j + \sum_{k=1}^{3} \lambda_{kj} x_k, \qquad j = 1, 2, 3,$$

is a rigid transformation, then

$$A'_{j,\rho}(u, v) = a_j + \sum_{k=1}^{3} \lambda_{kj} A_{k,\rho}(u, v), \qquad j = 1, 2, 3,$$

where

$$A'_{j,\rho}(u, v) \equiv \frac{1}{\pi\rho^2} \int\int_{\xi^2+\eta^2 \leqq \rho^2} x'_j(u + \xi, v + \eta) d\xi d\eta, \qquad j = 1, 2, 3.$$

2.2. S_ρ is invariant under each of the reflections

$$u' = u, \; v' = -v,$$

and

$$u' = -u, \; v' = v.$$

If, for example,

$$x'_j(u', v') \equiv x_j(u', -v'), \qquad j = 1, 2, 3,$$

then

$$A'_{j,\rho}(u', v') = A_{j,\rho}(u', -v'), \qquad j = 1, 2, 3,$$

where

$$A'_{j,\rho}(u', v') \equiv \frac{1}{\pi\rho^2} \int\int_{\xi^2+\eta^2 \leqq \rho^2} x'_j(u' + \xi, v' + \eta) d\xi d\eta, \qquad j = 1, 2, 3. \tag{6}$$

2.3. Similarly, S_ρ is invariant under rigid transformations in the (u, v)-plane.

2.4. Under the transformation

$$u' = \alpha u, \qquad v' = \alpha v, \qquad \alpha > 0, \tag{7}$$

the mean-value surface S_ρ is transformed into the mean-value surface $S'_{\alpha\rho}$ whose coordinate functions are given by (6), where

$$x'_j(u', v') \equiv x_j\left(\frac{u'}{\alpha}, \frac{v'}{\alpha}\right), \qquad j = 1, 2, 3.$$

Hence for α fixed, the *family* of mean-value surfaces $[S_\rho]$ associated with a given surface S is identical with the *family* of mean-value surfaces whose coordinate functions are given by (6); under the transformation (7), each member of one family is congruent to a member of the second family.

3. **Conformal mean-value surfaces.** Since, by Theorem 1.2, the only smooth surfaces in isothermic representation for which all associated mean-value surfaces are given in isothermic representation by (4) are (a) spherical surfaces, in representation whereby circles are mapped on circles, and (b) minimal surfaces, the question arises as to the nature of the mean-value surfaces in these two cases.

THEOREM 3.1. *If the functions* (1) *map a finite simply connected domain D isothermically on a minimal surface S, then each mean-value surface S_ρ associated with S is a minimal surface given in isothermic representation by* (4) *and coinciding with S for (u, v) in D_ρ.*

PROOF. By a theorem of Weierstrass,[4] the functions (1) are harmonic in D; consequently it follows from the mean-value property of harmonic functions that the functions (4) coincide with the functions (1) in the open set D_ρ. Hence all mean-value surfaces associated with minimal surfaces given in isothermic representation by (1) are themselves minimal surfaces given in isothermic representation by (4); the surface S_ρ coincides with S for (u, v) in D_ρ.

THEOREM 3.2. *If the functions* (1) *are not identically constant and if they map a finite simply connected domain D isothermically on a surface S that lies on a sphere $\mathfrak{S}$ of finite non-null radius a, such that circles are mapped on circles, then each mean-value surface S_ρ associated with S lies on a surface of revolution T_ρ and is given in isothermic representation by* (4). *Further, for $0<\rho<\infty$, T_ρ is not a sphere.*

PROOF. It has been pointed out, in a recently published paper,[5] that the functions (1) may be continued isothermically to map the entire closed u, v-plane isothermically on the whole of $\mathfrak{S}$; further, the functions (1) have the representation

[4] If the functions (1) are harmonic in D, and if (2) holds in D, then the functions (1) are said to form *a triple of conjugate harmonic functions*; see E. F. Beckenbach and T. Radó, *Subharmonic functions and minimal surfaces*, Transactions of this Society, vol. 35 (1933), pp. 648–661. Then the theorem of Weierstrass may be stated as follows. *A necessary and sufficient condition that the functions* (1), *defined in the domain D, be the coordinate functions of a minimal surface given in isothermic representation is that they form a triple of conjugate harmonic functions*; loc. cit., p. 649.

[5] Loc. cit., see Footnote 2; see p. 375.

$$
(8)\qquad
\begin{aligned}
x_1 &= x_1(u, v) \equiv a_1 + \frac{2a|k|\mathfrak{R}f(z)}{|f(z)|^2 + k^2},\\
x_2 &= x_2(u, v) \equiv a_2 + \frac{2a|k|\mathfrak{I}f(z)}{|f(z)|^2 + k^2},\\
x_3 &= x_3(u, v) \equiv k\left[1 - \frac{2a|k|}{|f(z)|^2 + k^2}\right],
\end{aligned}
$$

where $f(z)$ has one of the following forms:

$$
f(z) \equiv \alpha z + \beta, \qquad f(z) \equiv \alpha\bar{z} + \beta,
$$

where α and β are constants. In (8), $k = a_3 \pm a$, $|k|$ is the maximum of the two quantities $|a_3 + a|$ and $|a_3 - a|$ and (a_1, a_2, a_3) are the coordinates of the center of $\mathfrak{S}$.

From 2.1–2.3, it follows that we may assume $a_1 = a_2 = a_3 = 0$, $k = a$, in (8), and that we may assume the function $f(z)$ is given by

$$
(9)\qquad f(z) \equiv \alpha z, \qquad \alpha > 0.
$$

This is equivalent to assuming that the point $z = \infty$ corresponds to the point P: $(0, 0, a)$ on $\mathfrak{S}$, that the point $z = 0$ corresponds to the point P': $(0, 0, -a)$ which is diametrically opposite P, and that the point in the z-plane corresponding to $(a, 0, 0)$ is real and positive. Since we are investigating *all* mean-value surfaces associated with S, it follows from 2.4 that we may take $\alpha = 1$ in (9), in which case the functions (1) have the following familiar representation, as given by (8):

$$
(10)\qquad
\begin{aligned}
x_1 &= x_1(u, v) \equiv \frac{2a^2u}{u^2 + v^2 + a^2},\\
x_2 &= x_2(u, v) \equiv \frac{2a^2v}{u^2 + v^2 + a^2},\\
x_3 &= x_3(u, v) \equiv a\left[1 - \frac{2}{u^2 + v^2 + a^2}\right].
\end{aligned}
$$

If C_r is the circle $u^2 + v^2 = r^2$, and if

$$
z = u + iv = re^{i\theta},
$$

then (4) and (10) yield

$$
A_{1,\rho}(u, v) + iA_{2,\rho}(u, v) = e^{i\theta}[A_{1,\rho}(r, 0) + iA_{2,\rho}(r, 0)],
$$

$$
A_{3,\rho}(u, v) = A_{3,\rho}(r, 0),
$$

from which it follows that the map of C_r on the mean-value surface T_ρ, associated with the sphere $\mathfrak{S}$ which is defined by (10), is a circle C_r^* in a plane perpendicular to the x_3-axis; moreover, the center of C_r^* is on the x_3-axis. Since S_ρ lies on T_ρ, it follows that the functions (4) map D_ρ isothermically on a surface that lies on the surface of revolution T_ρ.

In §4 it will appear that for $0<\rho<\infty$, T_ρ is not a sphere. Nevertheless, we shall call T_ρ *a mean-value sphere.*

4. Mean-value spheres. The mean-value sphere T_ρ is a surface of revolution about the x_3-axis. Accordingly, to investigate T_ρ, it is sufficient to study the intersection T_ρ^* of T_ρ with the plane $x_2=0$. Since, by (10),

$$x_2(u, -v) \equiv -x_2(u, v),$$

it follows that the intersection of T_ρ with the plane $x_2=0$ can be obtained from (4) by setting $v=0$ in (10). A computation yields the following coordinate functions for T_ρ^*:

$$\begin{aligned} x_1 &= A_{1,\rho}(u, 0) \\ &\equiv 4a^2u/\left[a^2+\rho^2+u^2+((a^2+\rho^2-u^2)^2+4a^2u^2)^{1/2}\right], \\ x_3 &= A_{3,\rho}(u, 0) \\ &\equiv a - \frac{2a^3}{\rho^2}\log\frac{a^2+\rho^2-u^2+((a^2+\rho^2-u^2)^2+4a^2u^2)^{1/2}}{2a^2}. \end{aligned} \tag{11}$$

We make the following observations.

4.1. The curve T^* is symmetric about the x_3-axis; since

$$A_{1,\rho}(u, 0) \equiv A_{1,\rho}\left(\frac{u^2+\rho^2}{u}, 0\right),$$

$$A_{3,\rho}(u, 0) \equiv 2a - \frac{2a^3}{\rho^2}\log\frac{a^2+\rho^2}{a^2} - A_{3,\rho}\left(\frac{u^2+\rho^2}{u}, 0\right),$$

it follows that T_ρ^* is also symmetric about the line

$$x_3 = A_{3,\rho}((a^2+\rho^2)^{1/2}, 0) = a - \frac{a^3}{\rho^2}\log\frac{a^2+\rho^2}{a^2}. \tag{12}$$

From this symmetry and from the relations,

$$m = \frac{2au}{a^2+\rho^2-u^2}, \qquad \frac{dm}{du} = \frac{2a(a^2+\rho^2+u^2)}{(a^2+\rho^2-u^2)^2},$$

satisfied by the slope m, it follows that T_ρ^* is convex.

4.2. The height h and the width l of T_ρ^* are given by

$$h = \frac{2a^3}{\rho^2}\log\frac{a^2+\rho^2}{a^2}, \qquad l = \frac{4a^2}{(a^2+\rho^2)^{1/2}+a};$$

hence *for* $0<\rho<\infty$ *the curve* T_ρ^* *is not a circle.* Moreover, since

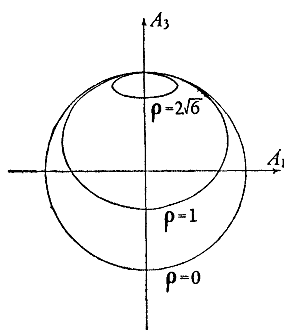
$$\lim_{\rho\to\infty}\frac{l}{h} = \infty,$$

it follows that T_ρ^* "flattens out" while approaching the point $(0, a)$, as $\rho\to\infty$.

4.3. Each member of the family $[T_\rho^*]$ passes through the point $(0, a)$ and is tangent there to every other member of the family. For $\rho=0$, T_ρ^* is the circle $x_1^2+x_3^2=a^2$, and for $\rho=\infty$, T_ρ^* is the point $(0, a)$.

4.4. A computation shows that for $0\leqq\rho<\rho'$, $T_{\rho'}^*$ is inside T_ρ^*, except for their common point of tangency.

The figure shows T_ρ^* for $\rho=0,\ 1,\ 2(6)^{1/2},\ \infty;\ a=1$.



From 4.1–4.4 it follows that the family of surfaces $[T_\rho]$ consists of convex surfaces each of which passes through the point $(0, 0, a)$ in (x_1, x_2, x_3)-space and is tangent there to every other member of the family. The surface T_ρ is a surface of revolution about the x_3-axis and is symmetric with respect to the plane

$$x_3 = a - \frac{a^3}{\rho^2}\log\frac{a^2+\rho^2}{a^2}.$$

Since the ratio of width to height $\to\infty$ as $\rho\to\infty$, it follows that T_ρ "flattens out" as $\rho\to\infty$. For $\rho=0$, T_ρ is the sphere about the origin with radius a, and for $\rho=\infty$, T_ρ is the point $(0, 0, a)$. For $0\leqq\rho'<\rho$, T_ρ is inside $T_{\rho'}$, except for their common point of tangency. From (11) we obtain the following isothermic representation for T_ρ:

$$x_1 = A_{1,\rho}(u, v)$$
$$\equiv \frac{a^2 u}{\rho^2(u^2+v^2)} \left[a^2+u^2+v^2+\rho^2-((a^2+\rho^2-u^2-v^2)^2+4a^2(u^2+v^2))^{1/2}\right],$$
$$x_2 = A_{2,\rho}(u, v)$$
$$\equiv \frac{a^2 v}{\rho^2(u^2+v^2)} \left[a^2+u^2+v^2+\rho^2-((a^2+\rho^2-u^2-v^2)^2+4a^2(u^2+v^2))^{1/2}\right],$$
$$x_3 = A_{3,\rho}(u, v)$$
$$\equiv a - \frac{2a^3}{\rho^2} \log\left[\frac{a^2-u^2-v^2+\rho^2+((a^2+\rho^2-u^2-v^2)^2+4a^2(u^2+v^2))^{1/2}}{2a^2}\right].$$

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NOTE ON THE DISTRIBUTION OF VALUES OF THE ARITHMETIC FUNCTION $d(m)$[1]

M. KAC

1. **Introduction.** Recently Dr. Erdös and the present writer[2] proved the following theorem:

If $\nu(m)$ denotes the number of different prime divisors of m and $k_n(\omega)$ the number of positive integers $m \leqq n$ for which

$$\nu(m) \leqq \lg \lg n + \omega(2 \lg \lg n)^{1/2},$$

then

$$\lim_{n\to\infty} \frac{k_n(\omega)}{n} = \pi^{-1/2} \int_{-\infty}^{\omega} e^{-u^2} du = D(\omega).$$

The purpose of this note is to derive a similar theorem concerning the function $d(m)$ which denotes the number of all different divisors of m (1 and m are included).

In fact we are going to prove the following theorem:

If $r_n(\omega)$ denotes the number of positive integers $m \leqq n$ for which

$$d(m) \leqq 2^{\lg \lg n + \omega(2 \lg \lg n)^{1/2}},$$

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[1] Presented to the Society, May 2, 1941.

[2] P. Erdös and M. Kac, *The Gaussian law of errors in the theory of additive number theoretic functions*, American Journal of Mathematics, vol. 62, pp. 738–742.