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ON THE DEFINITION OF CONTACT TRANSFORMATIONS

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If z is a function of $x_1, \cdots, x_n$ and $p_\nu = \partial z/\partial x_\nu$, $\nu = 1, \cdots, n$, a *contact transformation* in the space of $z, x_1, \cdots, x_n$, is defined by a set of $n+1$ equations

$$\text{(a)} \qquad Z = Z(z, x_\mu, p_\mu), \qquad X_\nu = X_\nu(z, x_\mu, p_\mu), \qquad \nu = 1, \cdots, n,$$

such that *firstly* in calculating the n derivatives

$$P_\nu = \frac{\partial Z}{\partial X_\nu}, \qquad \nu = 1, \cdots, n,$$

the expressions for the P_ν are given by a set of n equations

$$\text{(b)} \qquad P_\nu = P_\nu(z, x_\mu, p_\mu), \qquad \nu = 1, \cdots, n,$$

in which the derivatives of the p_μ *fall out*; and *secondly* the equations (a) and (b) can be resolved with respect to z, x_μ, p_μ:

$$\text{(A)} \qquad z = z(Z, X_\mu, P_\mu), \qquad x_\nu = x_\nu(Z, X_\mu, P_\mu), \qquad \nu = 1, \cdots, n,$$

$$\text{(B)} \qquad p_\nu = p_\nu(Z, X_\mu, P_\mu), \qquad \nu = 1, \cdots, n.$$

These two postulates are equivalent with the hypothesis that the $2n+1$ equations (a), (b) form a transformation between the two spaces of the sets of $2n+1$ independent variables (z, x_ν, p_ν), (Z, X_ν, P_ν) satisfying the Pfaffian condition

$$dZ - \sum_{\nu=1}^{n} P_\nu dX_\nu = \rho\left(dz - \sum_{\nu=1}^{n} p_\nu dx_\nu\right), \qquad \rho \neq 0.$$

In the following lines we prove: *the hypothesis that the system* (A) *is a corollary of the system* (a) *and conversely is already sufficient in order that* (a) *define a contact transformation*, that is to say: under this hypothesis the expressions (b) of P_ν, derived from (a), are independent of the second derivatives of z.

As to the functions $Z(z, x_\mu, p_\mu)$, $X_\nu(z, x_\mu, p_\mu)$, $z(Z, X_\mu, P_\mu)$, $x_\nu(Z, X_\mu, P_\mu)$, we shall assume:

(1) that the functions $Z(z, x_\mu, p_\mu)$, $X_\nu(z, x_\mu, p_\mu)$ possess continuous partial derivatives of the first order with respect to their $2n+1$ arguments;

(2) that the "total Jacobian"

$$\left| \frac{dX_\nu}{dx_\mu} \right|, \qquad \nu, \mu = 1, \cdots, n, \tag{1}$$

does not vanish identically in the $(2n+1)+n(n+1)/2$ variables z, x_ν, p_ν, $p'_{\nu x_\mu}$. Here the "*total derivative*" with respect to x_ν is defined by

$$\frac{d}{dx_\nu} = p_\nu \frac{\partial}{\partial z} + \frac{\partial}{\partial x_\nu} + \sum_{\mu=1}^{n} p'_{\mu x_\nu} \frac{\partial}{\partial p_\mu}, \qquad \nu = 1, \cdots, n; \tag{2}$$

(3) that the functions $z(Z, X_\mu, P_\mu)$, $x(Z, X_\mu, P_\mu)$ possess continuous partial derivatives of the first order with respect to their $2n+1$ arguments. (This hypothesis is certainly satisfied if the functions $Z(z, x_\mu, p_\mu)$, $X_\nu(z, x_\mu, p_\mu)$ possess continuous partial derivatives of the *second* order with respect to their arguments and if the determinant (1) does not vanish.)

From these three hypotheses it follows at once that the determinant $|dx_\nu/dX_\mu|$, ν, $\mu = 1, \cdots, n$, does not vanish identically, since $x_1, \cdots, x_n$ can be assumed as being independent variables.

Then, if $Z(z, x_\mu, p_\mu)$, $X_\nu(z, x_\mu, p_\mu)$ were all free of the p_μ, we have obviously a reversible point-to-point transformation between the space of $n+1$ variables (z, x_ν) and that of $n+1$ variables (Z, X_ν). And the same result holds if $z(Z, X_\mu, P_\mu)$, $x_\nu(Z, X_\mu, P_\mu)$ were all free of the P_μ. We may therefore assume without loss of generality that p_μ do actually appear in the equations (a) and P_μ in the equations (A).

By means of total derivatives (2), P_μ can be calculated from the n equations

$$\frac{dZ}{dx_\nu} = \sum_{\mu=1}^{n} P_\mu \frac{dX_\mu}{dx_\nu}, \qquad \nu = 1, \cdots, n. \tag{3}$$

Consider the n expressions

$$B_\nu = \frac{\partial Z}{\partial p_\nu} - \sum_{\mu=1}^{n} P_\mu \frac{\partial X_\mu}{\partial p_\nu}, \qquad \nu = 1, \cdots, n, \tag{4}$$

and suppose first that not all B_ν vanish.

Then, if for instance $B_1 \neq 0$, let

$$q_\lambda = \frac{\partial p_\lambda}{\partial x_1} = \frac{\partial p_1}{\partial x_\lambda}, \qquad \lambda = 1, \cdots, n.$$

In differentiating (3) with respect to q_λ we have easily

$$\sum_{\mu=1}^{n} P'_{\mu q_\lambda} \frac{dX_\mu}{dx_\nu} = \overset{\lambda}{\delta}_\nu B_1 + \overset{1}{\delta}_\nu (1 - \overset{1}{\delta}_\lambda) B_\lambda, \qquad \nu, \lambda = 1, \cdots, n,$$

where as usual

$$\overset{\mu}{\delta}_\nu = \begin{cases} 0, & \mu \neq \nu, \\ 1, & \mu = \nu. \end{cases}$$

But now it follows that

$$\frac{\partial(P_1, \cdots, P_n)}{\partial(p'_{1x_1}, \cdots, p'_{nx_1})} \left| \frac{dX_\mu}{dx_\nu} \right| = \left| \overset{\lambda}{\delta}_\nu B_1 + \overset{1}{\delta}_\nu (1 - \overset{1}{\delta}_\lambda) B_\lambda \right| = B_1^n \neq 0,$$

the P_ν are independent with respect to $p'_{1x_1}, \cdots, p'_{nx_1}$, and the equations (A) are only possible, if they do not contain the P_ν at all, the case which has been already discarded.

We have therefore $B_\nu = 0$, $\nu = 1, \cdots, n$. Then the equations (3) and (4) reduce to the $2n$ equations

$$\begin{aligned} \frac{\partial Z}{\partial x_\nu} + p_\nu \frac{\partial Z}{\partial z} &= \sum_{\mu=1}^{n} P_\mu \left(\frac{\partial X_\mu}{\partial x_\nu} + p_\nu \frac{\partial X_\mu}{\partial z} \right), \qquad \nu = 1, \cdots, n, \\ \frac{\partial Z}{\partial p_\nu} &= \sum_{\mu=1}^{n} P_\mu \frac{\partial X_\mu}{\partial p_\nu}, \qquad \nu = 1, \cdots, n. \end{aligned} \tag{5}$$

On the other hand, the rank of the matrix with n columns and $2n$ rows

$$\begin{pmatrix} \dfrac{\partial X_\mu}{\partial x_\nu} + p_\nu \dfrac{\partial X_\mu}{\partial z} \\ \dfrac{\partial X_\mu}{\partial p_\nu} \end{pmatrix}, \qquad \mu, \nu = 1, \cdots, n,$$

is n, since otherwise (1) would vanish. We see that in this case P_ν *can be expressed from* (5) *by* z, x_μ, p_μ.

Since the same argument applies to the equations (A), p_ν can be expressed by means of Z, X_μ, P_μ.

We have now the 4 sets of relations (a), (b), (A), (B). It is easily seen that the $2n+1$ relations (A), (B) are inverse of the $2n+1$ relations (a), (b), if p_μ resp. P_μ are considered as independent variables. Indeed, in putting the values (a) and (b) in the relations (A), (B), we must obtain identities $z=z$, $x_\nu=x_\nu$, $p_\nu=p_\nu$, for otherwise a non-identical relation between z, x_μ, p_μ would follow, that is, a differential equation, satisfied by an "arbitrary" function $z(x_1, \cdots, x_n)$.

We see that in the case of *one* function of n variables a reversible transformation of the first order is necessarily a contact transformation.

Our implicit definition of the "reversible transformations of the first order" leads to non-trivial results in the cases in which the contact transformations in the usual sense do not exist at all. For instance, in the case of $n>1$ functions $z_1(x), \cdots, z_n(x)$ of one independent variable, all contact transformations reduce simply to the point-to-point transformations in the space of $n+1$ variables $z_1, \cdots, z_n, x$. On the other hand, there exist in this case non-trivial reversible transformations. If for instance

$$X = z_n - x\sum_{\nu=1}^{n} p_\nu, \qquad Z_\lambda = z_\lambda, \quad (\lambda = 1, \cdots, n-1), \qquad Z_n = -\sum_{\nu=1}^{n} p_\nu,$$

$$p_\nu = \frac{dz_\nu}{dx}, \qquad P_\nu = \frac{dZ_\nu}{dX}, \qquad \nu = 1, \cdots, n,$$

we have easily for $\lambda = 1, \cdots, n-1$

$$\frac{P_\lambda}{xP_n - 1} = \frac{p_\lambda}{\sum_{\kappa=1}^{n-1} p_\kappa}, \qquad x = \frac{1 + \sum_{\lambda=1}^{n-1} P_\lambda}{P_n},$$

and therefore

$$z_n = X - \frac{Z_n}{P_n}\left(1 + \sum_{\lambda=1}^{n-1} P_\lambda\right), \qquad z_\lambda = Z_\lambda, \qquad \lambda = 1, \cdots, n-1.$$

We have determined in the case of n functions of one variable all reversible transformations of the first order by means of certain Pfaffian and Mongeian relations. These results will be exposed in another paper.

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