By the interior is meant all points of the plane which are not on the curve but which are separated from the point $\rho=0$ by the curve. By the exterior is meant all other points of the plane not on the curve nor in the interior of the curve.

THEOREM 3. *If, with the hypothesis of Theorem 2, the $2q$ radial lines through $\rho=0$ and the roots of $f'(z)$ and midway between the roots are drawn so that the plane is divided into $2q$ sectors, numbered from 1 to $2q$, then either all the roots of $f(z)$ must lie on the radial lines or there must be roots in an odd as well as in an even numbered sector.*

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ON THE REPRESENTATIONS, $N_7(m^2)$[1]

C. D. OLDS

1. **Introduction.** Write $N_r(n)$ for the number of representations of the positive integer n as the sum of r squares, and write $N_r(n, k)$ for the number of representations of n as the sum of r squares in which the first k squares in each representation are odd with positive roots, while the remaining $r-k$ squares are even with roots positive, negative, or zero. In a previous paper the author [5][2] gave an arithmetical derivation of the formula for $N_3(n^2)$. The method used to prove this result was based upon that employed by Hurwitz [2] in his discussion of the analogous formula for $N_5(n^2)$.

In 1930, G. Pall [6] gave an analytical derivation of the formula for $N_7(cn^2)$, c an integer. His formula shows, in particular, that if $m=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}$, where $p_1, p_2, \cdots, p_s$ are distinct odd primes, then

$$N_7(m^2) = 14\prod_{\nu=1}^{s}[p_\nu, \alpha_\nu], \tag{1}$$

where

$$[p_\nu, \alpha_\nu] = \sigma_5(p_\nu^{\alpha_\nu}) - (-1)^{(p_\nu-1)/2}p_\nu^2\sigma_5(p_\nu^{\alpha_\nu-1}).$$

We define the arithmetical function $\sigma_k(n)$, which occurs here, and the function $\rho_k(n)$, which occurs later, by the sums

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[1] This is the second part of a paper presented to the Society, April 6, 1940, under the title *On the number of representations of the square of an integer as the sum of an odd number of squares*. The author wishes to thank Professor J. V. Uspensky for help in preparing this paper.

[2] The numbers in brackets refer to the bibliography.

$$\sigma_k(n) = \sum_{d|n} d^k, \qquad \rho_k(n) = \sum_{n=d\delta} (-1)^{(\delta-1)/2} d^k, \qquad \delta \text{ odd},$$

where in the first sum d ranges over all divisors of n: $1 \leqq d \leqq n$, and in the second d, δ range over all integral solutions of the equation $n = d\delta$, where δ is odd. It is the purpose of this paper to show how formula (1) may be derived arithmetically. We shall use the method of Hurwitz [**2**].

2. **Preliminary formulas.** In the proof of (1) which follows, we use certain well known results concerning $N_r(n)$, when r is even. These formulas, which are not difficult to prove arithmetically,[3] are

$$N_6(m) = 12\rho_2(m), \qquad m \equiv 1 \pmod 4; \tag{2}$$

$$N_6(2m, 2) = \rho_2(m),\ N_6(2^{\alpha+2}m, 4) = 2^{2\alpha}\rho_2(m) = \chi(2^\alpha m),\ \alpha \geqq 0, m \text{ odd}; \tag{3}$$

$$N_{12}(4m, 4) + 16N_{12}(4m, 8) = \sigma_5(m), \qquad m \text{ odd}. \tag{4}$$

3. **Three lemmas.** The first lemma we need was proved by the author in [**5**].

LEMMA 1. *Let $f(n)$ be an arbitrary arithmetical function, not equal to zero for all n, and such that $f(nn') = f(n)f(n')$ for any two integers n and n'. If $F(n) = \sum f(d)$, where d ranges over all divisors of n, then*

$$F(nn') = \sum_{(a)} \mu(d) f(d) F(n/d) F(n'/d), \tag{5}$$

where (a) indicates that we take $d = 1, 2, 3, \cdots$, with the convention that $F(x) = 0$ if x is not an integer. Here $f(1) = F(1) = 1$ and $\mu(d)$ denotes the Möbius function.

Now let $P = N_7(m^2)$, m an odd integer, and let P_1 denote the number of solutions of the equation

$$m^2 = x^2 + y^2 + z^2 + t^2 + u^2 + v^2 + 4w^2,$$

and P_2 the number of solutions of the equation

$$m^2 = \bar{x}^2 + \bar{y}^2 + \bar{z}^2 + \bar{t}^2 + \bar{u}^2 + 4v^2 + 4w^2,$$

where the barred squares have odd roots and occupy the first five places in each representation. Then it may be proved without difficulty that

$$P = \tfrac{7}{6}(P_1 + 12P_2),$$

[3] Formula (2) may be found, for example, in J. Liouville [**3**], formulas (3) and (4) in Liouville [**4**]. For the method used to prove such formulas arithmetically, see T. Pepin [**7**] or P. Bachmann [**1**].

and thus our problem is reduced to the determination of convenient expressions for P_1 and P_2.

We first discuss P_1. To this end, notice that

$$P_1 = \sum_{(b)} N_6(m^2 - 4w^2),$$

where (b) indicates that $w=0, \pm 1, \pm 2, \cdots; 4w^2 < m^2$. On the other hand, (2) shows us that

$$N_6(m^2 - 4w^2) = 12\rho_2(m^2 - 4w^2),$$

so that

$$\begin{aligned} P_1 &= 12\sum_{(b)} \rho_2(m^2 - 4w^2) = 12\sum_{(b)} \rho_2((m - 2w)(m + 2w)) \\ &= 12\sum_{(c)} \rho_2(m'm''), \end{aligned}$$

where the last sum ranges over all positive odd integers m', m'' satisfying the equation (c): $2m = m' + m''$.

To evaluate $\rho_2(m'm'')$ use Lemma 1. Define

$$\begin{aligned} f(n) &= 0, \quad \text{whenever } n \text{ is even}, \\ f(n) &= (-1)^{(n-1)/2}n^2, \quad \text{whenever } n \text{ is odd}; \end{aligned}$$

then $f(nn') = f(n)f(n')$ for any two integers n, n' as required, and for odd n $(=m)$,

$$\begin{aligned} F(m) &= \sum_{d\mid m} f(d) = \sum_{d\mid m} (-1)^{(d-1)/2}d^2 = (-1)^{(m-1)/2}\sum_{m=d\delta} (-1)^{(\delta-1)/2}d^2 \\ &= (-1)^{(m-1)/2}\rho_2(m), \qquad \delta \text{ odd}. \end{aligned}$$

Applying (5) we obtain

$$\rho_2(m'm'') = \sum_{(a)} \mu(d)(-1)^{(d-1)/2}d^2\rho_2(m'/d)\rho_2(m''/d), \tag{6}$$

provided we adopt the convention that $\rho_2(x) = 0$ if x is not an integer. Hence

$$\begin{aligned} P_1 &= 12\sum_{(c)} \rho_2(m'm'') = 12\sum_{(c)}\sum_{(a)} \mu(d)(-1)^{(d-1)/2}d^2\rho_2(m'/d)\rho^2(m''/d) \\ &= 12\sum_{d=1,3,5,\cdots} \mu(d)(-1)^{(d-1)/2}d^2\sum_{(c)} \rho_2(m'/d)\rho_2(m''/d). \end{aligned}$$

Finally, with the aid of (3), we find

$$\sum_{(c)} \rho_2\left(\frac{m'}{d}\right)\rho_2\left(\frac{m''}{d}\right) = \sum_{(c)} N_6\left(\frac{2m'}{d}, 2\right)N_6\left(\frac{2m''}{d}, 2\right) = N_{12}\left(\frac{4m}{d}, 4\right).$$

Thus we have established

LEMMA 2. $P_1=12\sum_{d|m}\mu(d)(-1)^{(d-1)/2}d^2N_{12}(4m/d,\,4)$.

We evaluate P_2 in a similar manner. Since

$$P_2 = 2^4\sum_{(d)} N_6(m^2 - x^2,\,4),$$

where the extent of the summation is indicated by (d): $x=\pm 1, \pm 3, \cdots ; x^2<m^2$; we find, using (3), that

$$\begin{aligned} P_2 &= 16\sum_{(d)} \chi\left(\frac{m^2 - x^2}{4}\right) = 16\sum_{(d)} \chi\left(\frac{m+x}{2}\cdot\frac{m-x}{2}\right) \\ &= 16\sum_{(e)} \chi(n'n''), \end{aligned}$$

where n', n'' are any two positive integers satisfying the equation (e): $m=n'+n''$. Setting

$$n' = 2^\alpha m', \quad n'' = 2^\beta m'',$$

where m', m'' are odd, and using (6), we deduce that

$$\begin{aligned} \chi(n'n'') &= 2^{2(\alpha+\beta)}\rho_2(m'm'') \\ &= \sum_{(a)} \mu(d)(-1)^{(d-1)/2}d^2 2^{2\alpha}\rho_2\left(\frac{m'}{d}\right)2^{2\beta}\rho_2\left(\frac{m''}{d}\right) \\ &= \sum_{(a)} \mu(d)(-1)^{(d-1)/2}d^2\chi\left(\frac{n'}{d}\right)\chi\left(\frac{n''}{d}\right). \end{aligned}$$

As a result

$$P_2 = 16\sum_{(e)} \chi(n'n'') = 16\sum_{d=1,3,5,\cdots} \mu(d)(-1)^{(d-1)/2}d^2\sum_{(e)} \chi\left(\frac{n'}{d}\right)\chi\left(\frac{n''}{d}\right).$$

Now by (3)

$$\sum_{(e)} \chi\left(\frac{n'}{d}\right)\chi\left(\frac{n''}{d}\right) = \sum_{(e)} N_6\left(\frac{4n'}{d},\,4\right)N_6\left(\frac{4n''}{d},\,4\right) = N_{12}\left(\frac{4m}{d},\,8\right).$$

We have proved

LEMMA 3. $P_2=16\sum_{d|m}\mu(d)(-1)^{(d-1)/2}d^2N_{12}(4m/d,\,8)$.

4. **Derivation of the formula for $N_7(m^2)$.** Since we know already $P=\frac{7}{8}(P_1+12P_2)$, we find, using Lemma 2, Lemma 3, and finally (4), that

$$P = \frac{7}{6}\left[12 \sum_{d|m} \mu(d)(-1)^{(d-1)/2} d^2 N_{12}\left(\frac{4m}{d}, 4\right)\right.$$

$$\left. + 12\cdot 16 \sum_{d|m} \mu(d)(-1)^{(d-1)/2} d^2 N_{12}\left(\frac{4m}{d}, 8\right)\right]$$

$$= 14 \sum_{d|m} \mu(d)(-1)^{(d-1)/2} d^2 \left[N_{12}\left(\frac{4m}{d}, 4\right) + 16 N_{12}\left(\frac{4m}{d}, 8\right)\right]$$

$$= 14 \sum_{d|m} \mu(d)(-1)^{(d-1)/2} d^2 \sigma_5\left(\frac{m}{d}\right)$$

$$= 14 \prod_{\nu=1}^{s} [p_\nu, \alpha_\nu],$$

where

$$[p_\nu, \alpha_\nu] = \sigma_5(p_\nu^{\alpha_\nu}) - (-1)^{(p_\nu-1)/2} p_\nu^2 \sigma_5(p_\nu^{\alpha_\nu - 1}).$$

This is the result we set out to prove.

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