relative to positive definite quadratic forms and Diophantine equations. For two incommensurable collinear vectors, the euclidean algorithm becomes the continued fraction algorithm, a powerful tool in approximation problems. It is conjectured that the generalized algorithm is as effective for problems of simultaneous approximation as the continued fraction algorithm is for simple approximation problems. To support this conjecture, satisfactory solutions of such problems are obtained by use of the generalized algorithm. (Received April 24, 1941.)

300. Albert Whiteman: *Sums connected with the partition function.*

The sum $A_k(n) = \sum \exp(\pi i s(h, k) - 2\pi i h n/k)$, where h runs over a reduced residue system with respect to the modulus k and $s(h, k)$ is a Dedekind sum, appears in Rademacher's formula for the number of partitions of n. On the basis of a different expression for this sum Lehmer (Transactions of this Society, vol. 43 (1938), pp. 271–295) factored the $A_k(n)$ according to the prime number powers contained in k, and evaluated the $A_k(n)$ in the case in which k is a prime or a power of a prime. A new approach to the first of these results has recently been given in a paper by Rademacher and Whiteman (American Journal of Mathematics, vol. 63 (1941), pp. 377–407). In the present paper the second of these results is derived by a method which is considerably simpler than Lehmer's. The paper also contains a new method for evaluating certain generalized Kloosterman sums. (Received May 29, 1941.)

Analysis

301. R. P. Agnew: *On limits of integrals.*

It is shown that existence of $\lim_{A\to\infty}\int_A^{A+\lambda} f(t)dt$, for each λ in some set having positive measure, implies that the limit exists and is uniform over each finite interval of values of λ. The result is applied to prove two theorems of Iyengar (Proceedings of the Cambridge Philosophical Society, vol. 37 (1941), pp. 9–13) and the following Tauberian theorem. If $F(t)$ is absolutely continuous over each finite interval, if $\lim_{A\to\infty}\int_A^{A+\lambda} [F(t) - F'(t)]dt = 0$ for each real λ, and if $\lim_{t\to\infty} e^{-t}F(t) = 0$, then $\lim_{t\to\infty} F(t) = 0$. (Received April 11, 1941.)

302. Stefan Bergman: *A method for summation of series of orthogonal functions of two variables.*

Suppose $\{\Phi_\nu(\phi)\}$ $[(\phi) = (\phi_1, \phi_2)]$ is a system of O.N. functions $[0 \leqq \phi_k \leqq 2\pi$, $\Phi_k(\phi_1 + 2\pi, \phi_2) = \Phi_k(\phi_1, \phi_2 + 2\pi) = \Phi_k(\phi_1, \phi_2)$, $k = 1, 2]$, $\sigma(e)$ a completely additive set function, $f(\phi_1, \phi_2) \in L^{1+p}$ $(p > 0)$, $a_\nu = \int_0^{2\pi}\int_0^{2\pi} \Phi_\nu d\sigma$, $b_\nu = \int_0^{2\pi}\int_0^{2\pi} \Phi_\nu f d\phi_1 d\phi_2$. Consider the series (1) $\sum_{\nu=0}^{\infty} a_\nu \Phi_\nu(\phi_1, \phi_2)$ and (2) $\sum_{\nu=0}^{\infty} b_\nu \Phi_\nu(\phi_1, \phi_2)$. Let $\mathfrak{M}$ be a four-dimensional domain of the type described in Mathematische Annalen, vol. 104 (1931), pp. 611–636, with the distinguished boundary surface $\mathfrak{F} = \mathrm{E}[z_k = h_k(\phi_1, \phi_2), k = 1, 2]$. Let $\Psi_\nu(z_1, z_2)$, $\nu = 1, 2, \cdots$, be the functions of the extended class which assume the values Φ_ν on $\mathfrak{F}$, and (3) $S(z_1, z_2) = \sum_{\nu=0}^{\infty} a_\nu \Psi_\nu(z_1, z_2)$, (4) $F(z_1, z_2) = \sum_{\nu=0}^{\infty} b_\nu \Psi_\nu(z_1, z_2)$. The series (4) converges absolutely and uniformly in every closed subdomain of $\mathfrak{M}$ for $p > 1$. Using the results of Jessen, Marcinkiewicz and Zygmund (Fundamenta Mathematicae, vol. 25 (1935), pp. 217–234), Bergman and Marcinkiewicz (Fundamenta Mathematicae, vol. 33 (1939), pp. 75–94) and Bers (Comptes Rendus de l'Académie des Sciences, Paris, vol. 208 (1939), pp. 1273–1275 and 1475–1477) it is shown that S possesses a finite sectorial limit at the point $\{h_1(\phi), h_2(\phi)\}$ if σ possesses a finite strong derivative at (ϕ), and F possesses the sectorial limit f almost everywhere on $\mathfrak{F}$. (Received May 2, 1941.)

303. R. P. Boas: *Generalized Laplace integrals.*

The generalized Laplace integrals discussed have the form $f(z)=\int_0^\infty g(z, t)\phi(t)dt$, where $g(z, t)$ is in some sense "nearly" equal to the kernel e^{-zt}. The conditions imposed on $g(z, t)$ are analogous to those which have been imposed on functions $g_n(z)$ to make the series $\sum a_n g_n(z)$ behave like a power series (cf. Boas, Transactions of this Society, vol. 48 (1940), pp. 467–487). An application is made to the kernel $g(z, t)=(2zt/\pi)^{1/2}K_\nu(z, t)$ recently discussed by Meijer (Proceedings of the Section of Sciences, Koninklijke Akademie van Wetenschappen te Amsterdam, vol. 43 (1940), pp. 599–608, 702–711; see Mathematical Reviews, vol. 2 (1941), p. 96); here $-\frac{1}{2}<\Re(\nu)<\frac{1}{2}$, and $K_\nu(z)$ is the usual notation for a Bessel function of imaginary argument; when $\nu=\pm\frac{1}{2}$, $g(z, t)=e^{-zt}$. (Received May 19, 1940.)

304. D. G. Bourgin: *On reflexive Banach spaces.*

Several theorems are obtained generalizing a result of Ghantmacher and Smulian (Comptes Rendus de l'Académie des Sciences de l'URSS, vol. 17 (1937), p. 91). For instance: *If the unit sphere of E is sectionally compact of order $\aleph_a$ in the weak topology of E as elements and if E^* contains a dense set, of power less than or equal to $\aleph_a$, in the sense of the weak topology of E^* as functionals, then E is reflexive.* Some incidental results are also available. Thus a theorem implying the possibility of interchange of order of certain Moore-Smith limiting operations is: *If* (a) $\{f^{(l)}\}$ *is a $w^*(E^*)$ dense set of power less than or equal to $\aleph_a$ and* (b) $\{x^\rho\}$, *ρ a directed set of power less than or equal to $\aleph_a$, is contained in a $w(E)$ compact (of order $\aleph_a$) set, Q, and* (c) $f^l(x^\rho)$ *converges in the Moore-Smith sense to $f^l(x_0)$ for each l, then $\{x^\rho\}$ is $w(E)$ convergent to x_0 in the Moore-Smith sense.* (Received April 2, 1941.)

305. J. H. Curtiss: *Riemann sums and the fundamental polynomials of Lagrange interpolation.*

Let C denote an arbitrary Jordan curve of the z-plane and let $z=\phi(w)$ be an analytic function which maps the exterior K of C conformally onto the region $|w|>1$ so that the points at infinity correspond. Let $\phi'(\infty)=c$. Let $\omega_n(z)=\prod_{k=1}^n[z-\phi(e^{2\pi ik/n})]$, $n=1, 2, \cdots$. Then if C is rectifiable, $\lim_{n\to\infty}\omega_n(z)/(-c^n)=1$ uniformly for z on any closed point set of the region interior to C, and $\lim_{n\to\infty}\omega_n(z)/c^n(w^n-1)=1$ uniformly for z on any closed point set in K. Furthermore, the last equation is true uniformly for z in $C+K$ if $\phi'(w)$ is nonvanishing and of bounded variation for $|w|=1$. Auxiliary results used in the proof include an extension of a known result on Riemann sums and a proof that if $\phi'(w)$ is absolutely continuous for $|w|=1$, then $[\phi(w)-\phi(\bar{w})]/[w-\bar{w}]$ is uniformly absolutely continuous in either w or $\bar{w}$ for $|w|=1$. (Received April 18, 1941.)

306. M. H. Heins: *A generalization of the Aumann-Carathéodory "Starrheitssatz."*

Let F denote a Riemann surface in the sense of Weyl-Radó with the property that its fundamental group is non-abelian. Then the identical map $W\equiv w$ of F onto itself is isolated in the family of analytic maps $\{f(w)\}$ of F into itself in the sense that there exists no sequence of maps of the family $\{f_n(w)\}$ ($f_n\not\equiv w$, $n=1, 2, \cdots$) which converges pointwise to w as $n\to\infty$. The proof is based upon a study of the transforms of $f(w)$ with respect to a uniformizing map by means of Julia's lemma.

This theorem implies the following immediate corollaries: (a) the theorem of Klein and Poincaré concerning the proper discontinuity of the group of (1, 1) conformal maps of F onto itself where F is a surface of the above type; (b) the theorem of Koebe stating that the number of (1, 1) conformal maps of a plane region of finite connectivity greater than two onto itself is finite; (c) the "Starrheitssatz" of Aumann and Carathéodory. (Received April 2, 1941.)

307. Einar Hille: *Notes on linear transformations. III. Semi-groups on Lebesgue spaces.*

This paper deals with the properties of semi-groups on $L_1(-\pi, \pi)$ and $L_1(-\infty, \infty)$, essentially those which are multiplicative in the representation by Fourier series, Fourier transforms or Hermitian series. The first two types coincide with the semi-groups commuting with real translations. The discussion requires the solution of the factor problems of type (L, L) for Fourier transforms and for Hermitian series. Conditions are found for the convergence of T_α to T_0 in various topologies and the best degree of approximation is discussed. It is shown that any convex domain, invariant under addition, of the parameter plane can be the domain of existence of an analytic semi-group in $L_1(-\pi, \pi)$ and the rate of growth of $\|T_\alpha\|$ is studied when the domain is $\Re(\alpha)>0$. (Received May 5, 1941.)

308. Walter Leighton and W. J. Thron: *Convergence criteria for continued fractions.*

In the $z=x+iy$ plane let alternate coordinate axes x', y' be obtained by rotating the x, y axes through an arbitrary angle $0\leqq\beta<\pi/2$. Let $R(\beta)$ be the open convex region containing $z=0$ whose boundary is the broken line contour consisting of the two arcs of the parabola $y'^2=\cos\beta\cdot(x'+\frac{1}{4}\cos\beta)$ from infinity to the points of tangency of this parabola with the tangent rays drawn from the point $z=-\frac{1}{4}$ and completed by the segments of these tangent rays extending from $z=-\frac{1}{4}$ to the points of tangency. If the a_n are any complex numbers lying in any closed bounded region in $R(\beta)$, the continued fraction $1+K(a_n/1)$ converges. Further, for $\beta=\pi/2$ let R_1 be the open region defined by $-\frac{1}{4}<\Re(z)<0$, $\Im(z)>0$. If the a_n lie in a bounded closed region in R_1, the given continued fraction converges. In this theorem R_1 may be replaced by R_2: $-\frac{1}{4}<\Re(z)<0$, $\Im(z)<0$. When $\beta=0$ the region described becomes the parabola of Wall and Scott. (Received May 9, 1941.)

309. E. J. McShane: *Sufficient conditions for a weak relative minimum in the problem of Bolza.*

In the problem of Bolza there remains one large gap between the necessary conditions for a minimum and the sufficient conditions. For anormal problems, the condition on the positiveness of the second variation which figures in the sufficient conditions can not be established among the necessary conditions. Here a substitute is proposed for the condition of positiveness of the second variation which is equivalent for normal problems but weaker in anormal problems. It is shown that with this weakened assumption the sufficiency theorem for weak relative minima can still be established. It is conjectured that the new condition (with positiveness replaced by non-negativeness) can be shown to be necessary for a minimum. (Received April 3, 1941.)

310. J. D. Mancill: *On the solutions of a certain class of partial differential equations.*

It is shown in this note that the most general solution of the partial differential equation $(\sum_{i=1}^{n} x_i \partial/\partial x_i)^k F(x) = mF(x)$, $i=1, 2, \cdots, n$, where k is a positive integer and m is any constant, is the sum of p functions $F_j(x)$, $j=1, 2, \cdots, p$, $p \leqq k$, homogeneous of degree r_j but otherwise arbitrary; the r_j being the distinct roots of the equation $r(r-1)(r-2) \cdots (r-k+1) = m$. (Received May 19, 1941.)

311. E. N. Oberg: *Notes on the approximation of a function by sums of orthonormal functions.* Preliminary report.

Let $\phi_0, \phi_1, \phi_2, \cdots, \phi_n$ be a set of orthonormal functions on an interval (a, b), each having a derivative at every point. If $S_n(x) = a_0\phi_0 + a_1\phi_1 + \cdots + a_n\phi_n$ is an arbitrary sum of the ϕ's, it is shown that a bound for $|(d/dx)S_n(x)|$ exists in terms of $S_n(x)$. A study is made of the degree of convergence of sums of the form $S_n(x)$ to a function $f(x)$ on (a, b). Lastly, if $\Phi_n(x)$ is a sum of the ϕ's that minimizes the integral of $|f(x) - \Phi(x)|^p dx$, $p > 0$, then under certain hypotheses on $f(x)$, $\Phi_n(x)$ converges uniformly to $f(x)$ as $n \to \infty$. (Received April 7, 1941.)

312. George Polya and Norbert Wiener: *On the oscillation of the derivatives of a periodic function.*

Let $f(x)$ be periodic, real valued and infinitely derivable. Denote by N_k the number of changes of sign of the kth derivative of $f(x)$, in a period. Let $k \to \infty$. *If N_k is bounded, $f(x)$ is a trigonometric polynomial. If $N_k = o(k^{1/2})$, $f(x)$ is an entire function. If $N_k = O(k^{(1-\alpha)/2})$, where $0 < \alpha < 1$, $f(x)$ is an entire function of finite order not exceeding* $(1+\alpha^{-1})/2$. (Received April 19, 1941.)

313. W. T. Reid: *Green's lemma and related results.*

This paper is concerned with a proof of Green's lemma for the general case of a region R which is the interior of a simply closed rectifiable curve J, and where the continuity and differentiability conditions imposed on the functions involved are merely on $R+J$ and R, respectively. There is also established a general form of Cauchy's theorem for a function $f(z)$ which is holomorphic on R, and merely continuous on $R+J$. Finally, there is considered an associated result which is useful in the derivation of edge conditions for double integral problems of the calculus of variations. (Received April 11, 1941.)

314. M. S. Robertson: *The partial sums of multivalently star-like functions.*

Let $f(z) = z^p + a_{p+1}z^{p+1} + \cdots + a_n z^n + \cdots$ be regular and multivalently star-like of order p with respect to the unit circle $|z| < 1$. This means that within the unit circle $f(z)$ assumes no value more than p times, at least one value exactly p times, and in addition is star-like, that is, $\Re[zf'(z)/f(z)] > 0$, $|z| < 1$. When $p=1$, $f(z)$ is univalently star-like. The author shows that for $n > n_0(p)$ the nth partial sum $f_n(z) = z^p + a_{p+1}z^{p+1} + \cdots + a_n z^n$ is also multivalently star-like of order p for $|z| < 1 - A(p) \log n/n$ where $2p+1 \leqq A(p) \leqq 2p+2$. The order $\log n/n$ is best possible since the nth partial sum of the power series for $z^p(1-z)^{-2p}$ is multivalently star-like of order p in a circle about the origin whose radius is at most R_n where $R_n \geqq 1 - (2p+1) \log n/n$ and $\lim \sup_{n\to\infty}(1 - R_n/n^{-1} \log n) = 2p+1$. (Received April 3, 1941.)

315. L. B. Robinson: *Some complete systems of semitensors.*

The first problem is to find a complete system of semitensors of order one associated with the system of linear homogeneous differential equations (A) $y_i'' + \sum_{j=1}^{3} p_{ij} y_j' + \sum_{j=1}^{3} q_{ij} y_j = 0$ $(i = 1, 2, 3)$. The semitensors are transformed thus: $\bar{I} = \Delta_{1i} I_1 + \Delta_{2i} I_2 + \Delta_{3i} I_3$ where the Δ_{ji} are minors of the determinant of the transformation. Write $V_i \equiv D_{i1} I_1 + D_{i2} I_2 + D_{i3} I_3 = f_i$, in which the D_{ij} are minors of the determinant $|Z\ Y\ y|$ and the f_i arbitrary functions of the covariants of A. Solve with respect to the I_j and the complete system of semitensors results. To get semitensors of the second order, write $W_{ij} \equiv V_i \cdot V_j = f_{ij}$, wherein f_{ij} is an arbitrary function of the covariants, assume $I_i I_j \equiv I_{ij} \equiv I_{ji}$, solve with respect to I_{ij} and the complete system of semitensors is obtained. (Received April 7, 1941.)

316. W. J. Trjitzinsky: *Singular Lebesgue-Stieltjes integral equations.*

Integral equations, involving Stieltjes integrals, have been studied by a number of authors and, quite extensively, by N. Gunther. The kernels considered by Gunther are sufficiently "regular" to secure results resembling those of Fredholm and in the case of "symmetry," suitable defined, resembling those of Schmidt. The present work develops comprehensive theories of several types of integral equations, involving "symmetric" kernels essentially more general than those of Gunther; the kernels of the present work are representable as limits, in a suitable sense, of "regular" kernels. Our theory is based on Lebesgue-Stieltjes (Radon) integration, which appears to be an appropriate (in fact, necessary) tool for such problems. The developments are not of Fredholm type and they involve, in a very essential way, the use of an appropriate spectral theory. (Received April 3, 1941.)

Applied Mathematics

317. K. O. Friedrichs: *On the mathematical theory of the buckling of spherical shells.*

This paper is a contribution to von Kármán's recent nonlinear theory of buckling of spherical shells. The physically important state of equilibrium corresponds to a stationary value of the energy functional which is not a minimum but is of a degenerate character (no type number). The fact that buckling occurs only in the vicinity of one point is explained as a boundary layer phenomenon. For the analysis of the problem the Ritz method is employed. (Received April 2, 1941.)

318. F. G. Gravalos: *The algebraic integrals of Hill's equations.*

Hill's equations are $x'' - 2y' = 3x - \mu x/r^3$, $y'' + 2x' = -\mu y/r^3$, $r = (x^2 + y^2)^{1/2}$, μ a parameter. Such a system of differential equations admits of Jacobi's integral $\frac{1}{2}(x'^2 + y'^2) - \frac{3}{2}x^2 - \mu/r = c$. In this paper the theorem is proved: All the algebraic integrals of Hill's equations are functions of Jacobi's integral. The proof is done by parts in the form of three theorems. First, it is proved that all the integrals in the field $\Gamma(x', y', x, y, r)$ are reducible from Jacobi's. Second, the existence of integrals containing t rationally is ruled out. Finally, the above theorem is proved. Some parts of the proof are the same as those in a paper by C. L. Siegel on the restricted problem of three bodies (Transactions of this Society, vol. 39 (1936), pp. 225–237). (Received April 8, 1941.)