THE R_λ-CORRESPONDENT OF THE TANGENT TO AN ARBITRARY CURVE OF A NON-RULED SURFACE

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In a recent paper[1] the author defined at a general point y of a non-ruled analytic surface S the tangent line which he calls the *R_λ-correspondent* of the tangent at y to a general curve C_λ of S. It was proved[2] that (i) *a curve C_λ is a curve of Darboux if and only if at each of its points the R_λ-correspondent of the tangent to C_λ coincides with this tangent,* (ii) *a curve C_λ is a curve of Segre if and only if at each of its points the tangent to C_λ and its R_λ-correspondent are conjugate tangents of S.*

The primary purpose of this note is to present the following simple construction for the R_λ-correspondent: Let Λ denote a point of C_λ distinct from y, let U, V denote, respectively, the points of intersection of the asymptotic u- and v-curves passing through y with the asymptotic v- and u-curves passing through Λ, and let W denote the point of intersection of the tangent plane to S at y with the line joining the points U, V. *If y is held fixed while Λ tends toward y along C_λ, the point W describes a curve C_w and, except when C_λ is a curve of Segre or is tangent at y to a curve of Segre, the limit of W is the point y. The tangent at y to C_w is the R_λ-correspondent of the tangent to C_λ at y.*

The validity of this construction will be proved, and in addition the following theorem will be demonstrated:

A curve C_λ is a curve of Segre if and only if for a general point y of C_λ the limit of W as Λ tends to y along C_λ is a point W_0 distinct from y. The point W_0 is the intersection of the directrix of the first kind of Wilczynski with the tangent at y to the corresponding curve $C_{-\lambda}$ of Darboux.

Let the homogeneous projective coordinates $y^{(1)}, \cdots, y^{(4)}$ of a general point y on a non-ruled analytic surface S in ordinary space be functions of asymptotic parameters u, v. The functions $y^{(i)}$ are solutions of a system of differential equations, which can be reduced by a suitable transformation to Wilczynski's canonical form

$$(1) \qquad y_{uu} + 2by_v + fy = 0, \qquad y_{vv} + 2a'y_u + gy = 0.$$

The coefficients of these equations are functions of u, v which are connected by three conditions of integrability. Moreover, the coordinates

[1] P. O. Bell, *A study of curved surfaces by means of certain associated ruled surfaces.* Transactions of this Society, vol. 46 (1939), pp. 389–409.

[2] Loc. cit., p. 393.

$y^{(i)}$ are not solutions of any equation of the form $Ay_{uv}+By_u+Cy_v+Dy=0$ whose coefficients are functions of u, v not all zero. This statement implies that the point y_{uv}, whose coordinates are the functions $y_{uv}^{(i)}$, does not lie in the tangent plane to S at y.

An arbitrary one-parameter family F_λ of curves of S is defined by the curvilinear differential equation $dv-\lambda du=0$, where λ is an arbitrary function of u, v. We denote by C_λ the curve of F_λ which passes through y. If u, v be regarded as functions of a single parameter t, as t varies, the point y, whose curvilinear coordinates are u, v, describes a curve of S. This curve will be the curve C_λ if the functions $u=u(t)$, $v=v(t)$ are selected such that for a general value of t

$$\lambda(u, v) = v'/u', \tag{2}$$

where accents indicate differentiation with respect to t.

The curvilinear coordinates of the point Λ are given by $u(t+\overline{\Delta t})$, $v(t+\overline{\Delta t})$. The points U and V are therefore given by $u(t+\overline{\Delta t})$, v and u, $v(t+\overline{\Delta t})$. The general homogeneous coordinates of the points U and V are consequently functions of t, and may be represented by the developments

$$\begin{aligned} U = {} & y + y_u u'\overline{\Delta t} + (y_{uu}u'^2 + y_u u'' u')\overline{\Delta t}^2/2 \\ & + (y_{uuu}u'^3 + 3u'^2u''y_{uu} + f_1 y_u)\overline{\Delta t}^3/6 \\ & + (y_{uuuu}u'^4 + 6u'^2u''y_{uuu} + f_2 y_{uu} + f_3 u_u)\overline{\Delta t}^4/24 + \cdots, \end{aligned} \tag{3}$$

$$\begin{aligned} V = {} & y + y_v v'\overline{\Delta t} + (y_{vv}v'^2 + y_v v'' v')\overline{\Delta t}^2/2 \\ & + (y_{vvv}v'^3 + 3v'^2v''y_{vv} + g_1 y_v)\overline{\Delta t}^3/6 \\ & + (y_{vvvv}v'^4 + 6v'^2v''y_{vvv} + g_2 y_{vv} + g_3 y_v)\overline{\Delta t}^4/24 + \cdots \end{aligned} \tag{4}$$

wherein f_i, g_i, $i=1, 2, 3$, represent functions of u, v which for our purpose do not require explicit determination.

By differentiating equations (1) we find that the coefficients of y_{uv} in the expressions for y_{uuu}, y_{vvv}, y_{uuuu}, y_{vvvv} are $-2b$, $-2a'$, $-4b_u$, $-4a'_v$, respectively. The coefficients of y_{uv} in the expressions for the homogeneous coordinates of the points U, V are, therefore,

$$-\ bu'^3\overline{\Delta t}^3/3 - (b_u u'^4 + 3bu'^2u'')\overline{\Delta t}^4/6 + \cdots,$$

$$-\ a'v'^3\overline{\Delta t}^3/3 - (a'_v v'^4 + 3a'v'^2v'')\overline{\Delta t}^4/6 + \cdots,$$

respectively. The point W, which is the intersection of the tangent plane to S at y with the line joining U, V has homogeneous coordinates which may be obtained by forming a linear combination of those of U and V which contains no y_{uv} term. Hence, such a combination is

$$(2a'v'^3 + [a_v' v'^4 + 3a'v'^2v'']\overline{\Delta t})U - (2bu'^3 + [b_u u'^4 + 3bu'^2u'']\overline{\Delta t})V.$$

Expanding this we obtain the expression

$$(5)\qquad \begin{aligned} &2(a'v'^3 - bu'^3)y + (a_v' v'^4 + 3a'v'^2v'' - b_u u'^4 - 3bu'^2u'')\overline{\Delta t}y \\ &\qquad + 2(a'v'^3u'y_u - bu'^3v'y_v)\overline{\Delta t} + \text{terms of order } \overline{\Delta t}^2 \end{aligned}$$

for the homogeneous coordinates of the point W. If $a'v'^3 - bu'^3 \neq 0$, the limit of W as $\overline{\Delta t}$ tends to zero is, clearly, the point y. Moreover, the tangent to C_w at y has the direction defined by $dv/du = -bu'^2/a'v'^2$. This is the direction of the R_λ-correspondent of the tangent to C_λ at y. This completes the proof for the general case in which λ is not a direction of Segre.

The curve C_λ is a curve of Segre if and only if at each of its points the direction defined by $\lambda = v'/u'$ satisfies the equation $a'v'^3 - bu'^3 = 0$. In this case it is clear from (5) that the limit of W as $\overline{\Delta t}$ tends to zero is a point W_0, distinct from y, whose homogeneous coordinates are given by

$$(6)\qquad \begin{aligned} &(3a'v'^2v'' - 3bu'^2u'' + a_v v'^4 - b_u u'^4)y \\ &\qquad + 2(a'v'^3u'y_u - bu'^3v'y_v), \qquad \text{where } a'v'^3 = bu'^3. \end{aligned}$$

If we divide this expression by $a'u'v'^3$, make use of the condition $a'v'^3 = bu'^3$, and make the following substitutions, $v'/u' = \lambda$, $-v'u''/u'^3 = \lambda_u$, $v''/u'v' = \lambda_v$, we obtain the simpler form

$$(7)\qquad \begin{aligned} &(3[\lambda_u + \lambda\lambda_v]/\lambda + a_v'\lambda/a' - b_u/b)y \\ &\qquad + 2(y_u - \lambda y_v), \qquad \text{where } a'\lambda^3 = b, \end{aligned}$$

for the coordinates of W_0. It is, clearly, a simple matter to evaluate (7) explicitly for a direction $\lambda = \epsilon(b/a')^{1/3}$, wherein ϵ is a cube root of unity. The result is

$$(8)\qquad 2y_u - a_u' y/a' - \epsilon(b/a')^{1/3}(2y_v - b_v y/b).$$

This expression is a linear combination of the expressions $2y_u - a_u'y/a'$ and $2y_v - b_v y/b$ for the homogeneous coordinates of the points in which the directrix of the first kind intersects the asymptotic u- and v-tangents to S at y. Moreover, the ratio of the coefficient of y_v to that of y_u is the direction $-\epsilon(b/a')^{1/3}$ of Darboux which corresponds to the direction $\epsilon(b/a')^{1/3}$ of Segre. This completes the demonstration of the theorem.

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