ON REGULAR FAMILIES OF CURVES[1]

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A family F of non-intersecting curves filling a metric space is called *regular* if, in a neighborhood of any point p, it is homeomorphic with a family of straight lines. We have given in another paper[2] a necessary and sufficient condition, which we shall call (A′) (to be described below), that a family F be regular. We shall prove in this note that the following condition is sufficient:

(A) Given any point p, and a direction on the curve through p, there is an arc pq in this direction with the following property. For every $\epsilon > 0$ there is a $\delta > 0$ such that for any p', with $\rho(p', p) < \delta$, there is an arc $p'q'$ of $C(p')$ such that

$$p'q' \subset V_\epsilon(pq), \qquad q' \subset V_\epsilon(q). \tag{1}$$

The condition (A′) is the same, except that after (1), we add:

(2) If r' and s' are on $p'q'$ and $\rho(r', s') < \delta$, then $\delta(r's') < \epsilon$.

From the present theorem it is clear that the families of curves recently defined by Niemitzki[3] are regular.

To prove the theorem, suppose (A) holds, but (A′) does not. Then the following is true:

(B) There is a point p, and a direction of the curve $C(p)$, such that for any arc pq on $C(p)$ in this direction, there is an $\epsilon > 0$, such that for any $\delta > 0$, there is a point p', with $\rho(p', p) < \delta$, such that for any q' on $C(p')$,

(3) either $p'q' \not\subset V_\epsilon(pq)$, or $q' \not\subset V_\epsilon(q)$,

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[1] Presented to the Society, April 27, 1940.

[2] Annals of Mathematics, (2), vol. 34 (1933), pp. 244–270. We refer to this paper as RF. By RF, Theorem 7A, F is regular as there defined. The converse is proved as follows. By Theorem 17A, there is a cross-section S through p. In a neighborhood of p, the curves are orientable (this is easily seen, for instance, with the help of Theorem 9B). Choose an open subset S' of S, and let U be all points $q' = g'(q, \alpha)$, q in S', $|\alpha| < \epsilon$ (see RF, §15); U is a neighborhood of p, expressed as the product of S' and the open line segment $-\epsilon < \alpha < \epsilon$.

By a curve, we shall mean here the topological image of an open line segment or of a circle. We shall use $\rho(p, q)$ for distance, $\delta(A)$ for the diameter of the set A, and $V_\epsilon(A)$ for the set of all points p, $\rho(p, A) < \epsilon$. Let $C(p)$ mean the curve of F through p.

[3] V. Niemytzki, Recueil Mathématique de Moscou, vol. 6 (48) (1939), pp. 283–292. We mention two further papers in the subject: H. Whitney, Duke Mathematical Journal, vol. 4 (1938), pp. 222–226, showing that if the curves fill a region in 3-space, a cross-section may be chosen so as to be a 2-cell; W. Kaplan, Duke Mathematical Journal, vol. 7 (1940), pp. 154–185, studying families filling the plane.

(4) or there are points r', s' on $p'q'$ such that $\rho(r', s') < \delta$, and $\delta(r's') \geqq \epsilon$.

Choose a point p and a direction on $C = C(p)$, by (B). Choose q on C in this direction, by (A). Choose $\epsilon > 0$ by (B). For each positive integer i, choose δ_i by (A), with ϵ replaced by ϵ/i. Choose p_i by (B), with δ replaced by δ_i. Choose q_i by (A), with p' replaced by p_i; then

$$p_i q_i \subset V_{\epsilon/i}(pq), \qquad q_i \subset V_{\epsilon/i}(q). \tag{5}$$

By (B), as $\epsilon_i < \epsilon$, we may choose p_i' and q_i' on $p_i q_i$ so that

$$\rho(p_i', q_i') < \delta_i, \qquad \delta(p_i' q_i') \geqq \epsilon. \tag{6}$$

By (6), we may choose r_i on $p_i' q_i'$ so that $\rho(p_i', r_i) \geqq \epsilon/2$. By (5) and (6), we may choose a subsequence so that for some points p' and r on pq,

$$p'_{\lambda_i} \to p', \qquad q'_{\lambda_i} \to p', \qquad r_{\lambda_i} \to r; \tag{7}$$

then $r \neq p'$. Say, for definiteness, that r is in the direction of q from p'. The set of such points r which are limits of such sequences $\{r_{\lambda_i}\}$ forms a closed set, which, by (5), is in $p'q$; we shall let r be the point furthest from p'. (It might be q.)

Assuming that (A) holds for the point r and the direction away from p', we shall arrive at a contradiction. Choose a point s on C in this direction from r, by (A). (If C is a closed curve, it might happen that s is on the arc pr.) Choose r' and s' on C just behind and just in front of r, so that r' is on neither pp' nor rs, and s' is not on $p'r$. We shall show that for any $\epsilon' > 0$ there is an integer j and a point s_j on $p_j' q_j'$ within ϵ' of s'; as s is in pq, by (5), this will contradict the definition of r, and thus prove the theorem.

Set

$$4\eta = \min\,[\rho(r', rs), 2\epsilon']. \tag{8}$$

Choose r_-, r_+, s_-, s_+ on C in the order $r_- r' r_+ r s_- s' s_+ s$, so that r_- is not in rs and s_+ is not in pr (if C is closed), and so that

$$r_- r' r_+ \subset V_\eta(r'), \qquad s_- s' s_+ \subset V_\eta(s'). \tag{9}$$

Set

$$2\epsilon'' = \min\,[\rho(pr_-, r_+q), \rho(rs_-, s_+s), \eta]. \tag{10}$$

Using r, s, and ϵ'', choose $\delta'' > 0$ by (A). By (7), we may choose j so that

$$\epsilon/j < \epsilon'', \ \rho(p_j', p') < \epsilon'', \ \rho(q_j', p') < \epsilon'', \ \rho(r_j, r) < \delta''. \tag{11}$$

By the choice of δ'', we may choose s^* on $C(r_j)$ so that

$$r_j s^* \subset V_{\epsilon''}(rs), \qquad s^* \subset V_{\epsilon''}(s). \tag{12}$$

As $r_j s^*$ is a connected set, (11), (12) and (10) show that there is a point s_j on it such that $\rho(s_j, rs_- + s_+ s) \geqq \epsilon''$; hence, by (12), $\rho(s_j, s_- s_+) < \epsilon''$, and by (9),

$$\rho(s_j, s') < \epsilon'' + \eta < 2\eta \leqq \epsilon'. \tag{13}$$

By (12) and (8),

$$\rho(r', r_j s_j) > 2\eta. \tag{14}$$

By (5) and (11),

$$p_j' q_j' \subset V_{\epsilon''}(pq). \tag{15}$$

By (11), (10), (5) and (9), there are points p_j^* in $p_j' r_j$ and q_j^* in $r_j q_j'$ such that

$$\rho(p_j^*, r') < 2\eta, \qquad \rho(q_j^*, r') < 2\eta. \tag{16}$$

By this and (14), the arc $r_j s_j$ is contained in the arc $p_j^* r_j q_j^* \subset p_j' q_j'$. Hence $s_j \subset p_j' q_j'$, which, with (13), gives the contradiction.

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