62. W. D. Rannie: *Tensor methods in the theory of turbulence.* Preliminary report.

Applications of tensor analysis to the statistical theory of isotropic turbulence as developed by T. von Kármán and L. Howarth, and later treated by H. P. Robertson, are reviewed. (Received November 18, 1940.)

63. Eric Reissner: *A new derivation of the equations for the deformation of elastic shells.*

The equations of the theory of small deformations of shells, first given by A. E. H. Love, are rederived in a simpler manner. The simplifications are accomplished by using (1) vector stress resultants and equilibrium conditions in vector form and (2) the three-dimensional system of orthogonal coordinates which goes with the lines of curvature on the middle surface of the shell and the strain components with respect to this system. The assumption that the normal to the undeformed middle surface is deformed into the normal to the deformed middle surface, satisfied by determining appropriate displacement components, is introduced into these strain components. (Received November 25, 1940.)

64. H. J. Stewart: *Steady state oscillations in an atmosphere on a rotation sphere.* Preliminary report.

If one plots the mean surface atmospheric pressure, averaged over a period of at least a week, one finds that in addition to the mean westerly flow of air, there exist large scale closed isobaric systems which change very slowly with time. Attempts to develop long range weather forecasting techniques have shown the positions of these systems to be of primary importance and a knowledge of the factors which control these systems is very useful as a guide in formulating forecasting methods. In the present paper certain steady state oscillations of the stratosphere are investigated and are shown to vary with the mean velocity in the same manner as the observed oscillations. (Received November 18, 1940.)

## Geometry

65. P. O. Bell: *On differential geometry intrinsically connected with a surface element of projective arc length.*

In this paper a surface element of projective arc length is interpreted geometrically and used to obtain a new geometric interpretation for each of the following: a generalization of Bompiani's projective curvature, a generalization of Fubini's asymptotic curvature, a projective torsion introduced in this paper, conjugate tangents, the tangents of Darboux, and the tangents of Segre. The associate conjugate net of an arbitrary net $N_{\lambda_1\lambda_2}$ of a surface $S$ (introduced in this paper) is defined as the conjugate net whose tangents at a point $P$ of $S$ separate harmonically the tangents at $P$ of the net $N_{\lambda_1\lambda_2}$. The following characteristic property of this net is a typical result: Let arcs $PP_1$, $PP_2$ of equal projective length $s$ be measured, with respect to the form $ds=(2Rv')^{1/2}du$, from the point $P$ along the curves $C_{\lambda_1}$, $C_{\lambda_2}$, respectively, of the net $N_{\lambda_1\lambda_2}$. The tangent plane to $S$ at $P$ intersects the line joining $P_1P_2$ in a point $P_3$ which tends to a limit point $P_0$, distinct from $P$, as $s$ tends to zero. The tangent line joining $PP_0$ and its conjugate tangent envelop the conjugate associate of the net $N_{\lambda_1\lambda_2}$, as $P$ varies over $S$. (Received November 20, 1940.)

### 66. L. M. Blumenthal: *Betweenness in metric ptolemaic spaces.*

A metric ptolemaic space ($MP$-space) is a metric space for which the determinant $|(p_ip_j)^2|$, $(i, j=1, 2, 3, 4)$, is nonpositive for every quadruple of points $p_1$, $p_2$, $p_3$, $p_4$ of the space. Investigating such spaces with regard to the four-, three-, and two-triple properties shows: (1) each $MP$-space has the four-triple property; (2) an $MP$-space has the three-triple property unless it contains the vertices of a convex tripod (that is, four points with one of them between each two of the remaining three points); (3) if four points of an $MP$-space contain exactly two linear triples, they may be labelled $a$, $b$, $c$, $d$ so that either $abc$ and $abd$ or $acb$ and $adb$ hold. Thus an $MP$-space has the two-triple property unless it contains vertices of a fork or a bow. Besides the transitive property of the betweenness relation valid in all metric spaces, there is in $MP$-spaces the additional transitivity $abc$ and $bad \rightarrow cad$ and $abd$. These results are preliminary to a study of separable, complete, convex $MP$-spaces which contain for each three non-linear points at least one point equidistant from them. (Received November 23, 1940.)

### 67. L. M. Blumenthal and C. V. Robinson: *Helly theorems on the sphere.*

Theorems for families of convex subsets of the sphere are given corresponding to the following theorem due to E. Helly: If each $n+1$ members of a family of convex bodies of $E_n$ intersect, then all the members of the family have a common point. On the sphere the existence of a common point is implied by the intersection of each 3, 4, 5, or 6 sets of the family according to the generality of the family considered. (Received November 26, 1940.)

### 68. C. R. Cassity: *The double points of a pencil of cubics invariant under the quadratic transformation.*

The double points of a general pencil of cubics contained in the invariant web of cubics of the involutorial quadratic transformation lie at the invariant points of the transformation and by pairs on the four lines which contain the six base points of the pencil which are not base points of the web. (Received November 25, 1940.)

### 69. Nathaniel Coburn: *A note on conformal geometry.*

Two Riemannian spaces of $n$-dimensions $V_n$ and $'V_n$ are considered. If coordinate systems in $V_n(\xi^\lambda)$ and $'V_n(\xi^\lambda)$ can be chosen so that at corresponding points $P(\xi^\lambda)$, $'P(\xi^\lambda)$ the connections of the spaces are related by the conformal transformation, then are the spaces conformally related? The problem is shown to be equivalent to determining the number of independent solutions of a system of linear partial differential equations. By analyzing the integrability conditions of the system, it is shown that only one independent solution exists. Hence the spaces are necessarily conformal. The complete theorem is: If coordinate systems in $V_n(\xi^\lambda)$ and $'V_n(\xi^\lambda)$ exist so that at corresponding points $P(\xi^\lambda)$, $'P(\xi^\lambda)$ the connections of these spaces are related by the conformal transformation and if the principal directions of the metric tensor of $'V_n$ exist in $V_n$, then the spaces are conformal. This theorem is of interest in that it can be shown that the corresponding theorem is not valid in unitary spaces. (Received November 20, 1940.)

### 70. Nathaniel Coburn: *Unitary spaces with corresponding geodesics.*

In the first section of the paper, the equations of geodesic curves $X_1$, which depend on a real parameter ($t$) and which are imbedded in a unitary space of $n$-dimensions $K_n$, are derived from the calculus of variations (Euler equations). The principal result is: the equations of such geodesics differ from the equations of geodesics in Riemannian space in that the former contain the torsion affinor whereas the latter do not contain this affinor. In the second section, the discussion centers on the connections of two unitary spaces $K_n$ and $'K_n$ whose geodesics correspond. It is shown that: (1) if two unitary spaces, both with symmetric connections, have their geodesics in correspondence, then the connections are related by the projective transformations; (2) if $K_n$ has a connection with torsion and $'K_n$ has a symmetric connection, then their geodesics cannot correspond. The first result is obtained in the same manner as the similar result in Riemannian space. The problem of determining all connections of unitary spaces $K_n$, both with torsion, whose geodesics correspond is left open. (Received November 20, 1940.)

### 71. Richard Courant: *Critical points and unstable minimal surfaces.*

Morse and Tompkins, and independently Shiffman (Annals of Mathematics, (2), vol. 40 (1939), pp. 834–854), have shown that Morse's theory of critical points in function spaces can be applied to minimal surfaces spanning suitably smooth contours. The main difficulty to overcome is the proof of a deformation property of the Dirichlet functional in the space of harmonic vectors. For this purpose a thorough analysis of the explicit Douglas boundary functional or its equivalent is needed. The present note, in line with the author's previous work on the Douglas problem, attacks the question for polygonal contours in a wider space by intrinsic considerations. It solves the problem by reducing it to that of the stationary points of a function of a finite number of variables with continuous derivatives. (Received November 27, 1940.)

### 72. N. A. Court: *On the harmonic pole.*

If $U$, $V$ are two tetrahedrons (triangles) polar reciprocal for a quadric (conic), and $L$ is the pole of a plane $\lambda$ (line $l$) for the quadric (conic), the harmonic plane (line) of $L$ for $U$ and the harmonic pole of the plane $\lambda$ (line $l$) for $V$ are pole and polar plane (line) with respect to the quadric (conic). Various consequences of this proposition are considered, of which the following may be noted: The harmonic plane (line) of a point of a quadric (conic) for a tetrahedron (triangle) inscribed in the quadric (conic), and the harmonic pole, for the tangential tetrahedron (triangle), of the tangent plane (line) to the quadric (conic) at the point considered are polar for the quadric (conic). (Received November 23, 1940.)

### 73. N. A. Court: *On the skew cubic.*

Given a tetrahedron $T$ whose faces osculate a skew cubic $C_3$ the harmonic pole for $T$ of a variable osculating plane of $C_3$ lies on a fixed line $j$. Conversely, given $T$ and $j$, the harmonic plane for $T$ of a variable point of $j$ osculates a skew cubic. The transforms of $j$ in the three skew harmonic homologies having for axes the three pairs of opposite edges of $T$ are axes of the cubic, and so are the transforms of $j$ in the four homologies, of constant $-3$, of which the vertices and the respectively opposite faces of $T$ are the centers and planes of homology. Various properties connected with these axes of $C_3$ are considered. (Received November 23, 1940.)

**74. S. B. Myers:** *Complete Riemannian manifolds of positive mean curvature.*

The author has proved previously that if, on a complete $n$-dimensional Riemannian manifold $M$, the curvature at every point and with respect to every pair of directions is greater than a fixed positive constant, then $M$ is closed (compact) and so is its universal covering manifold. In the present paper the same conclusions are drawn from the weaker hypothesis that the mean curvature of $M$ at every point and with respect to every direction is greater than a fixed positive constant. In particular, a complete space of constant positive mean curvature is closed, and so is its universal covering manifold. Such spaces are important in the general theory of relativity. (Received November 25, 1940.)

## Statistics and Probability

**75. G. A. Baker:** *Fundamental distributions of errors for agricultural field trials.*

Evidence from various sources is presented which shows that the fundamental error distribution for yield trials is represented by $[1/ab(t_1-t_0)]\int_0^a\int_0^b\int_{t_0}^{t_1}[1/\sigma(x, y, t)] \exp -\frac{1}{2}\{[\xi-f(x, y, t)]^2/\sigma^2(x, y, t)\}dtdydx$ where the integrals may be Stieltjes integrals. Under certain conditions the fundamental error distribution can be expressed as a Gram-Charlier series, but very rarely, if ever, as a normal distribution. For comparison with analysis of variance results based on the normal theory, the distribution of the ratio of independent estimates of the second moments of samples, if the fundamental distributions are Gram-Charlier series, are given. Similar considerations show that the distributions of the numbers attacked in field trials can rarely be represented by Poisson or binomial distributions as is usually assumed. (Received October 22, 1940.)

**76. G. A. Baker:** *Maximum likelihood estimation of the ratio of the components of nonhomogeneous populations.*

Let $f(x)=[1/(1+k)](f_1(x)+kf_2(x))$, $e\leqq x\leqq f$, $k>0$ and $k<\infty$, where $f_1(x)$ and $f_2(x)$ are probability functions. The problem is to find the maximum likelihood estimate of $k$, say $\bar{k}$. If $f_1(x)$ and $f_2(x)$ are rectangular with equal ranges that partially overlap, then the probability of a value of $\bar{k}=w/u$ (where $u$ is the number of individuals drawn from the nonoverlapped interval of $f_1(x)$, $w$ is the number of individuals drawn from the nonoverlapped interval of $f_2(x)$ and $v$ is the number of individuals drawn from the interval overlapped by $f_1(x)$ and $f_2(x)$) is $(n!/u!v!w!)(p_1)^u(p_2)^v(p_3)^w$ where the $p_i$'s are the probabilities of coming from the respective intervals. The cases for which $u=n$, $v=n$, $w=n$, $u=0$, $w=0$ are excluded because $\bar{k}$ is then indeterminate. Hence, the probability of a determinate value of $\bar{k}$ is $P=1+(p_2)^n-(p_2+p_3)^n-(p_1+p_2)^n$. The estimates of $k$ are biased. (Received October 22, 1940.)

**77. G. F. McEwen:** *Statistical problems of the range divided by the mean in samples of size n.*

Certain quantitative climatological studies are based upon the "precipitation ratio" or ratio to the mean annual rainfall of the difference between the maximum and minimum annual rainfall corresponding to the standard number of years. Available observations correspond to various values of the number of years $n$. Accordingly it is necessary to compute the precipitation ratio $I$ corresponding to a standard number