ON UNCONDITIONAL CONVERGENCE IN NORMED VECTOR SPACES

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Suppose X is a complete normed vector or Banach space of elements x. Orlicz[1] has given the following two definitions of unconditional convergence of an infinite series $\sum_n x_n$ of elements from X and proved their equivalence:

A. $\sum_n x_n$ is unconditionally convergent if and only if any rearrangement of the series is convergent.

B. $\sum x_n$ is unconditionally convergent if and only if $\sum_k x_{n_k}$ converges, where $\{x_{n_k}\}$ is any subsequence of $\{x_n\}$.

Pettis[2] has shown that either of these conditions is equivalent to the statement:

C. Every subseries of $\sum_n x_n$ is weakly convergent to an element of X, that is, $\{x_{n_k}\}$ implies the existence of an element x_σ such that, for every $\bar{x}$ of the conjugate space $\overline{X}$, $\sum_k \bar{x}(x_{n_k}) = \bar{x}(x_\sigma)$.

In proving this equivalence, he shows that condition C implies the following:

D. $\lim_n \sum_{m=n}^{\infty} \bar{x}(x_m) = 0$ uniformly for $\|\bar{x}\| = 1$.

E. H. Moore[3] has shown that for real, complex or quaternionic numbers, absolute and therefore unconditional convergence is equivalent to the following definition of convergence:

Let σ be any finite subset $n_1, \cdots, n_k$ of the positive integers, and denote $\sum_{i=1}^{k} x_{n_i}$ by $\sum_\sigma x_n$. Then

E. $\sum x_n$ converges in the σ-sense, if $\lim_\sigma \sum_\sigma x_n$ exists, where the limit is the Moore-Smith limit, and $\sigma_1 \geqq \sigma_2$ means that σ_1 contains all of the numbers in σ_2.[4]

Obviously the Moore-Smith limit can be extended to normed vector spaces, and the fundamental properties carry over. It is the purpose of this note to show that convergence in the σ-sense is equivalent to each of the conditions A, B, and D, that is, A, B, D and E are equivalent definitions of unconditional convergence, to which the

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[1] *Ueber unbedingte Konvergenz in Funktionenräumen*, Studia Mathematica, vol. 4 (1933), pp. 33–38.

[2] *Integration in vector spaces*, Transactions of this Society, vol. 44 (1938), pp. 281–282.

[3] *General Analysis*, Memoirs of the American Philosophical Society, vol. 1, part 2, 1939, p. 63.

[4] See Alaoglu, Annals of Mathematics, (2), vol. 41 (1940), p. 259, where a similar definition for weak unconditional convergence is given.

Pettis result adds C as a fifth equivalent definition. We consider definition E the most elegant of the definitions of unconditional convergence.[5]

A *is equivalent to* E. Assume that every rearrangement of $\sum x_n$ is convergent and suppose if possible $\lim_\sigma \sum_\sigma x_n$ is not equal to $x=\sum_n x_n$, summed in its natural order. Then there exists an $e>0$, such that for every σ there exists a $\sigma'\geqq\sigma$ such that $\|x-\sum_{\sigma'}x_n\|\geqq e$. Let n_0 be chosen so that for $n\geqq n_0$ we have $\|x-\sum_1^n x_m\|<e/2$. Let σ_1 be the set $1, 2, \cdots, n_0$, and σ_1' chosen so that $\|x-\sum_{\sigma_1'}x_n\|\geqq e$. Let σ_2 include all of the integers less than or equal to any integer in σ_1'. Repeating the process produces a σ_2', a σ_3, and so on, and defines a rearrangement of $\sum x_n$, namely, $\sigma_1, \sigma_1'-\sigma_1, \sigma_2-\sigma_1', \cdots$ (where $\sigma-\sigma'$ means the elements of σ not in σ'), which is not a convergent series since

$$\left\|\sum_{\sigma_m} x_n - \sum_{\sigma_m'} x_n\right\| = \left\|\sum_{\sigma_m'-\sigma_m} x_n\right\| \geqq e/2.$$

Since $\lim_\sigma \sum_\sigma x_n$ is unique when it exists, it follows at once that every rearrangement of $\sum_n x_n$ converges to the same limit.[6]

Conversely, suppose $\sum_\sigma x_n$ approaches a limit x, that is, for every $e>0$, there exists a σ_e such that if $\sigma\geqq\sigma_e$ then $\|\sum_\sigma x_n - x\|\leqq e$. Let $\{x_{n_k}\}$ be any rearrangement of $\{x_n\}$. Then we need only to choose k_e so that the set of integers $n_1, \cdots, n_{k_e}$ includes all of the integers in σ_e to be assured that for $k'\geqq k_e$ it is true that

$$\left\|\sum_{k=1}^{k'} x_{n_k} - x\right\| \leqq e.$$

B *is equivalent to* E. Suppose every subseries $\sum_k x_{n_k}$ converges but $\lim_\sigma \sum_\sigma x_n$ does not exist, that is, there exists an $e>0$ such that for every σ there exist $\sigma', \sigma''\geqq\sigma$ such that $\|\sum_{\sigma'}x_n-\sum_{\sigma''}x_n\|>e$. If $\sigma'+\sigma''$ denotes the set including all elements of σ' and σ'', then either $\|\sum_{\sigma'}x_n-\sum_{\sigma'+\sigma''}x_n\|\geqq e/2$ or $\|\sum_{\sigma''}x_n-\sum_{\sigma'+\sigma''}x_n\|\geqq e/2$, that is, we can assume $\sigma'\geqq\sigma''\geqq\sigma$. We obtain a nonconvergent subseries as follows: Take $\sigma_1=1$. This gives rise to $\sigma_1'\geqq\sigma_1''\geqq\sigma_1$ so that $\|\sum_{\sigma_1'}x_n-\sum_{\sigma_1''}x_n\|>e$. Let $n_1, \cdots, n_{k_1}$ be the elements of $\sigma_1'-\sigma_1''$. Take $\sigma_2=\sigma_1'$, and $n_{k_1+1}, \cdots, n_{k_2}$ to be the elements of $\sigma_2'-\sigma_2''$. Proceeding in this manner we get a series $x_{n_1}+x_{n_2}+\cdots+x_{n_{k_1}}+\cdots+x_{n_{k_2}}+\cdots$ such that $\|\sum_{k_m+1}^{k_{m+1}}x_{n_k}\|>e$, that is, one which is not convergent.

[5] In view of the results of Orlicz and Pettis, it would be sufficient to prove that D implies E implies A. For the sake of completeness and elegance we have preferred to prove each equivalence separately.

[6] See Orlicz, Studia Mathematica, vol. 1 (1929), p. 242.

Conversely if $\lim_\sigma \sum_\sigma x_n$ exists and $\{x_{n_k}\}$ is any subsequence of x_n, then if k_e is chosen so that n_{k_e} is larger than any integer appearing in the σ_e involved in the definition of the limit, it will be certain that for $k' \geqq k_e$ and any l

$$\left\| \sum_{k=k'}^{k'+l} x_{n_k} \right\| \leqq 2e,$$

that is, $\sum_k x_{n_k}$ is convergent.

D *is equivalent to* E. Suppose $\lim_n \sum_{m=n}^\infty |\bar{x}(x_m)| = 0$ uniformly for $\|\bar{x}\| = 1$. We demonstrate that $\lim_{\sigma_1,\sigma_2} \|\sum_{\sigma_1} x_n - \sum_{\sigma_2} x_n\| = 0$. This is obviously equivalent to showing that $\lim_{\sigma_1 \geqq \sigma_2} \|\sum_{\sigma_1} x_n - \sum_{\sigma_2} x_n\| = 0$, since

$$\left\| \sum_{\sigma_1} x_n - \sum_{\sigma_2} x_n \right\| \leqq \left\| \sum_{\sigma_1} x_n - \sum_{\sigma_1+\sigma_2} x_n \right\| + \left\| \sum_{\sigma_2} x_n - \sum_{\sigma_1+\sigma_2} x_n \right\|.$$

Let n_e be such that for $n \geqq n_e$ we have $\sum_n^\infty |\bar{x}(x_m)| \leqq e$ if $\|\bar{x}\| = 1$, and take $\sigma_e = 1, 2, \cdots, n_e$ with $\sigma_1 \geqq \sigma_2 \geqq \sigma_e$. Then $\sum_{\sigma_1 - \sigma_2} |\bar{x}(x_n)| \leqq e$ and so $|\bar{x}(\sum_{\sigma_1-\sigma_2} x_n)| \leqq e$. Since this is uniform for $\|\bar{x}\| = 1$, it follows that

$$\left\| \sum_{\sigma_1 - \sigma_2} x_n \right\| = \text{l.u.b.} \left[\left| \bar{x}\left(\sum_{\sigma_1-\sigma_2} x_n \right) \right| \text{ for } \|\bar{x}\| = 1 \right] \leqq e.$$

Conversely suppose $\lim_\sigma \sum_\sigma x_n$ exists. Then for $\sigma_1 \geqq \sigma_2 \geqq \sigma_e$ we have $\|\sum_{\sigma_1-\sigma_2} x_n\| \leqq e$. Take n_e greater than the largest integer in σ_e. Let $m \geqq n_e$ and $\|\bar{x}\| = 1$. Denote by $\sigma_{1\bar{x}}$ those of the integers $m, m+1, \cdots, m+k$, for which $\bar{x}(x_n) \geqq 0$, and by $\sigma_{2\bar{x}}$ the integers for which $\bar{x}(x_n) < 0$. Then

$$\begin{aligned} \sum_m^{m+k} |\bar{x}(x_n)| &= \sum_{\sigma_{1\bar{x}}} |\bar{x}(x_n)| + \sum_{\sigma_{2\bar{x}}} |\bar{x}(x_n)| \\ &= \left| \bar{x}\left(\sum_{\sigma_{1\bar{x}}} (x_n) \right) \right| + \left| \bar{x}\left(\sum_{\sigma_{2\bar{x}}} x_n \right) \right| \leqq \left\| \sum_{\sigma_{1\bar{x}}} x_n \right\| + \left\| \sum_{\sigma_{2\bar{x}}} x_n \right\| \leqq 2e; \end{aligned}$$

that is, $\lim_n \sum_n^\infty |\bar{x}(x_n)| = 0$, uniformly for $\|\bar{x}\| = 1$. A similar procedure would take care of the case in which $\bar{x}(x)$ were complex valued.

We note that condition D is equivalent to the statement that the linear operations (functionals) $\bar{x}$ for which $\|\bar{x}\| = 1$ map the sequence x_n on a compact subset of l, the space of absolutely convergent series, that is, any unconditionally convergent series can be interpreted as a completely continuous transformation on the adjoint space $\overline{X}$ to the space l.[7]

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[7] See Dunford, Transactions of this Society, vol. 44 (1938), p. 322.

Further it can be deduced from condition B that if $x(\sigma)=\sum_k x_{n_k}$ where $\sigma=n_1, n_2, \cdots$ then $x(\sigma)$ is a completely additive set function on subsets of the integers to the space X.

E. H. Moore, in the reference mentioned above, considers the case in which the set of integers $1, 2, \cdots, n, \cdots$ is replaced by a general set: $\mathfrak{P}$, and then defines a general sum $\sum x(p)$ by an obvious generalization. He shows that if $\sum x(p)$ exists, then $x(p)$ is zero except at a denumerable set of elements $p_1, \cdots, p_n, \cdots$ and $\sum_n |x(p_n)|$ exists. This result is extensible to the case where x is on $\mathfrak{P}$ to a linear normed complete space in the form:

If x is on $\mathfrak{P}$ to X and if $\sum x(p)$ exists in the sense that $\lim_\sigma \sum_\sigma x(p)$ exists, where the σ are finite subsets of $\mathfrak{P}$, then $x(p)$ differs from zero at most at a denumerable set of elements $p_1, \cdots, p_n, \cdots$ and $\sum x(p_n)$ is unconditionally convergent.

The first part of this theorem depends on the fact, easily derived by a slight change in the proof of "E implies D" above, that if $\sum_p x(p)$ exists then $\lim_\sigma \sum_\sigma |\bar{x}(x_p)|$ exists uniformly for $\|\bar{x}\|=1$, that is, for every $e>0$ there exists a σ_e such that if $\sigma_1 \geqq \sigma_2 \geqq \sigma_e$ and $\|\bar{x}\|=1$ then $\sum_{\sigma_1-\sigma_2} |\bar{x}(x_p)| \leqq e$. Let $\mathfrak{P}_0$ be the sum of the sets σ_e for $e=1/n$. This set will be denumerable. If p is not of $\mathfrak{P}_0$ then $|\bar{x}(x_p)| \leqq 1/n$ for all n and $\|\bar{x}\|=1$, that is, $x_p=0$. The second part of the theorem is obvious.

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