ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

446. Warren Ambrose: *Change of velocities in a continuous ergodic flow.* Preliminary report.

A flow is a 1-parameter group T_t, $-\infty < t < \infty$, of measure preserving transformations of a space into itself. It is measurable (continuous) if T_tP is a measurable (continuous) (P, t)-function If S is any measure preserving transformation of a space Ω into itself, then a flow can be "built on S" as follows: consider the product space of Ω with $0 \leqq x < 1$, with measure defined multiplicatively. Then define the flow by $T_t(P, x) = (S^nP, t+x-n)$, where n is equal to the integral part of $t+x$. It is shown that for any continuous ergodic flow on a separable metric space of finite measure the velocities along the streamlines can be altered to obtain a flow built on a measure preserving transformation, that is, a subspace of the original space can be found whose product with $0 \leqq x < 1$ is, with respect to measure properties, the original space (where measure on that original space is now the measure invariant under the altered flow) and such that the altered flow is built on a measure preserving transformation on this subspace. It is intended to use this theorem in a study of spectral properties which are invariant under a change of velocities. (Received August 5, 1940.)

447. Salomon Bochner and I. J. Schoenberg: *On positive definite functions on compact spaces.*

The expansion theorem for positive definite functions on finite-dimensional euclidean spheres, as stated in *On positive definite functions on spheres* by I. J. Schoenberg (abstract 46-11-474), is contained in a general theorem concerning such functions on compact spaces on which a transitive group of transformations is defined. In the general case, as in the special case, the function is given as a function of two independent points which in addition to being positive definite is invariant under simultaneous transformations of both points by the same group element. (Received August 5, 1940.)

448. D. G. Bourgin and Benjamin Epstein: *A class of kernels generated by a Laplace-Mellin transformation.*

The kernels discussed include those of the type of the generalized zeta function. The authors treat the inversion in L_2 of integrals with these kernels. (Received August 5, 1940.)

449. G. W. Brown: *Reduction of certain composite statistical hypotheses.*

The results obtained make it possible to reduce a large class of "composite" statistical hypotheses to equivalent "simple" hypotheses. The fundamental theorem established states essentially that if two distributions give rise, in sampling, to the same distribution of the set of differences between observations, then one distribution must be a translation of the other, subject to a condition requiring that the characteristic function of one of the distributions be such that any interior intervals of zeros be not too large. The result is established by means of the functional equation $\phi(t_1)\phi(t_2)\phi(-t_1-t_2)=\psi(t_1)\psi(t_2)\psi(-t_1-t_2)$ relating the characteristic functions. Similar results are obtained for scale, and combination of location and scale, and the corresponding situations in multivariate distributions. This type of uniqueness theorem permits one to reduce a composite statistical hypothesis involving an unknown location parameter (or scale, or both) to an equivalent simple hypothesis. (Received August 2, 1940.)

450. Leonard Carlitz: *On the Staudt-Clausen theorem.*

The principal object of this note is to simplify both the proof and formulation of the "polynomial" analogue of the Staudt-Clausen theorem given some time ago (Duke Mathematical Journal, vol. 3 (1937), pp. 503–517). (Received August 6, 1940.)

451. H. B. Curry: *The inconsistency of certain formal logics.*

In 1935 Kleene and Rosser published a proof that certain systems of formal logic were inconsistent in the sense that every formula which could be formulated in them was also demonstrable. This proof has been revised and simplified by the present author, who showed that the inconsistency depends, essentially, on the conjunction of two incompatible properties, called combinatorial and deductive completeness respectively (see abstract 43-3-118). The argument, in both the original proof and the revision, was a refinement of the Richard paradox. In the present paper it is shown that a contradiction can be derived by a much simpler argument based on the paradox of Epimenides. (Received August 6, 1940.)

452. Samuel Eilenberg: *Local connectedness in higher dimensions.*

A metric space Y which is locally p-connected in the homotopy sense for $k \leqq p \leqq n$ (notation: Y is LC_k^n) has the following property (P): Given any metric separable space X and a closed subset X_1 which is LC_0^{k-1} such that $\dim X - X_1 \leqq n+1$, every continuous mapping $f(X_1) \subset Y$ has an extension $f(U) < Y$ where U is some open set containing X_1. If Y is locally 0-connected then the property (P) is equivalent with Y being LC_k^n. If the property (P) holds without the restriction $\dim X - X_1 \leqq n+1$ then Y is said to be an absolute neighborhood retract starting from the dimension k (notation: Y is ANR_k). If Y is compact then Y is an absolute neighborhood retract if and only if Y is LC_0^{k-1} and ANR_k. A compact space Y which is ANR_1 may not be locally p-connected for any $p \geqq 0$; however, if Y is locally 0-connected then it is LC_0^n for every n. (Received August 9, 1940.)

453. Samuel Eilenberg: *On homotopy groups.*

Let Y be an arcwise connected topological space. A singular simplex in Y is a couple (G^p, f) where G^p is a closed euclidean p-simplex and f is a mapping $f(G^p) \subset Y$. After a proper equivalence relation has been introduced, the totality of all the singular cells in Y was shown by Lefschetz to form a closure-finite complex $S(Y)$. The nth homology group $H^n(S(Y))$ coincides with the homology group $H^n(Y)$ obtained using singular chains and cycles. Given $y_0 \varepsilon Y$ and $m \geqq 0$ let $S_m(Y)$ be the subcomplex of $S(Y)$ consisting of all singular simplexes (G^p, f) such that every face of G^p of dimension less than

m is mapped by f into y_0. For every n the groups $H^{n,m}(Y)=H^n(S_m(Y))$ are considered where $0\leqq m\leqq n$. The groups $H^{n,0}(Y)$, $H^{n,1}(Y)$ and $H^n(Y)$ are isomorphic. If $n>1$ and the coefficients are integers, then $H^{n,n}(Y)$ is the nth homotopy group $\pi_n(Y)$ of Y. (Received August 9, 1940.)

454. C. H. Forsyth: *Statistical interpretations of the first auxiliaries of the Gaussian symmetrical method.*

While the Gaussian symmetrical method of applying least squares is well known and fully treated in mathematical literature, the fact that the first auxiliaries of that method form a veritable treasure house for most of the fundamental formulas of statistical theory (including those of means, dispersions and even of correlation coefficients) has not, apparently, been duly appreciated. The author applies the notation peculiar to statistical theory to discover all these formulas. (Received August 7, 1940.)

455. Hilda P. Geiringer: *A generalization of the law of large numbers.*

Let $V_1(x)$, $V_2(x)$, $\cdots$, $V_n(x)$ be n probability distributions which are not supposed to be independent and let $F(x_1, \cdots, x_n)$ be a "statistical function" of n observations in the sense of von Mises, $V_i(x)$, $i=1, \cdots, n$, indicating as usual the probability of getting a result equal to or less than x at the ith observation. Then it can be proved that under fairly general conditions $F(x_1, \cdots, x_n)$ converges stochastically to its theoretical value; or, in other words, that under these general conditions a great class of statistics $F(x_1, \cdots, x_n)$ are consistent in the sense of R. A. Fisher. Well known particular cases of this theorem result (a) if $F(x_1, \cdots, x_n)$ is taken for the average $(x_1+x_2+\cdots+x_n)/n$ of the n observations, (b) if it is assumed that the $V_i(x)$ are independent distributions. (Received August 5, 1940.)

456. D. W. Hall and W. T. Puckett: *Strongly arcwise connected sets.*

In this paper two characterizations of strongly arcwise connected (see W. T. Puckett, this Bulletin, abstract 46-5-316) sets are obtained. The first characterization states that a locally connected continuum A is strongly arcwise connected if and only if every infinite collection of open sets in A contains an infinite subcollection intersecting some arc of A. Strong arcwise connectivity of a set at a point p is defined and a necessary and sufficient condition that the locally connected continuum M fail to be strongly arcwise connected at a point p is obtained. From this criterion it is deduced that an arbitrary locally connected continuum M is strongly arcwise connected at all save possibly a countable number of its points. If no two points separate M, then M is strongly arcwise connected. (Received September 11, 1940.)

457. D. C. Harkin: *Fourier series as limits of Gaussian sums.*

By means of the cyclotomic algebra which Gauss introduces in the seventh section of the *Disquisitiones Arithmeticae*, Fourier series can be had as limiting cases of sums similar to those there considered. (Received August 15, 1940.)

458. O. G. Harrold (National Research Fellow): *Characterizations of some continua by continuous functions.* Preliminary report.

In this paper a study is made of mappings of a given continuum into the interval (0, 1) which are characteristic. This method has been used by Mazurkiewicz, Whyburn and Eilenberg. New characterizations are given for the continua of finite degree and the continua having no continuum of condensation. For the last mentioned class of continua, a subclass of all continuous mappings is found with respect to which this type of continuum has much the same properties as the class of all continuous mappings with respect to the class of all Peano continua. If a mapping of A into B is called of *almost all finite sections* provided that $f^{-1}(y)$ is finite, $y \varepsilon B$, except for a countable set of points each of which has a countable inverse (A.F.S.), then a necessary and sufficient condition that a Peano continuum admit an A.F.S. map into (0, 1) is that each dendrite in the continuum have this property. This is related to a problem proposed by Cech (Fundamenta Mathematicae, vol. 18). (Received August 6, 1940.)

459. M. H. Heins: *A proof of Picard's theorem by the fundamental theorem on the conformal mapping of simpy-connected Riemann surfaces.*

Let G_w^∞ denote the universal covering surface of G_w, the extended w-plane with three distinct points a, b, c deleted. G_w^∞ is simply-connected, and, as the universal covering surface of the Riemann surface G_w, G_w^∞ may be taken to be a Riemann surface. Hence G_w^∞ can be mapped one-to-one and conformally onto one and only one of the following canonical domains: (1) the extended z-plane, (2) the finite z-plane, (3) the interior of the unit circle in the z-plane. The first possibility is readily excluded. As for the second, let $w=w(z)$ denote a mapping function which allegedly maps the finite z-plane one-to-one and conformally onto G_w^∞. $w=w(z)$ would be automorphic under a group of linear transformations mapping the finite z-plane onto itself. From this one infers readily that $w(z)$ must be doubly-periodic, simply-periodic, or a constant. In each case a contradiction is manifest. Hence G_w^∞ must be of hyperbolic type. Picard's theorem that a function, not a constant, meromorphic in the finite plane, can omit at most two values follows immediately. (Received August 6, 1940.)

460. L. C. Hutchinson: *On the classification of the trivector.*

In this paper the author finds by a new method the known canonical forms for trivectors, or alternating tensors of the third valency, in eight dimensions, as well as new forms which do not exist as canonical forms except in the real domain, and examines some of their geometrical properties. This paper is based on the author's work for the doctorate at the Massachusetts Institute of Technology. (Received August 6, 1940.)

461. Nathan Jacobson: *Restricted Lie algebras of characteristic p.*

In this paper certain identities previously noted connecting pth powers, $a+b$, $a\alpha$, α in the field and $[a, b]=ab-ba$ in an associative algebra of characteristic $p\neq 0$, are proved to be characteristic of these operations. This is shown by defining the concept of an abstract restricted Lie algebra and proving that any such algebra L may be obtained as a subset of an associative algebra closed under the above operations. Among the minimal associative algebras determined by L in this way, there is one which is universal in a certain sense. The derivation algebra of any algebra of characteristic p is an instance of a restricted Lie algebra. A notion of a restricted derivation of a restricted Lie algebra is defined. These form a restricted Lie algebra and may be used to define a restricted holomorph. If L has a finite basis it has a (1-1) representation by finite matrices. An analogue of Engel's theorem holds in this case. (Received August 9, 1940.)

462. R. L. Jeffrey: *Integration in abstract space.*

A theory of integration in a complete normed vector space is given which is shown to be equivalent to that given by G. Birkhoff (Transactions of this Society, vol. 38 (1935), pp. 357–378). This theory is in the spirit of the classical theories for real and complex variables. In particular it makes no use of convex sets, and, as a consequence of this, the developments are as simple as those of the classical theories. (Received August 3, 1940.)

463. F. B. Jones: *Quasi-continuous curves and certain boundary problems.*

A continuum M is said to be *quasi-connected at* P if P belongs to M and for each point X of $M-P$ there exists a continuum which lies in $M-X$ and contains an open subset of M containing P. A continuum which is quasi-connected at each of its points is said to be a *quasi-continuous curve.* The class of quasi-continuous curves includes as proper subclasses both the class of continuous curves and the class of semi-locally connected continua, whereas neither of the latter classes includes the other. The first part of the paper deals largely with the similarity between quasi-continuous curves and continuous curves in a Hausdorff space. The second part of the paper deals with boundary problems in a space whose only two-dimensional feature is the Jordan curve theorem. If P is a point of a continuum K in such a space so that (1) K is locally compact at P and (2) K is quasiconnected at P, then K together with all but a finite num ber of its complementary domains is connected im kleinen at P. As an interesting application of this theorem, it is shown that a proposition analogous to Whyburn's generalization of the Torhorst theorem for the plane holds true in these abstract spaces. (Received August 10, 1940.)

464. S. C. Kleene: *Recursive predicates and quantifiers.* Preliminary report.

Consider the following forms of expression: $R(n)$, $(Ex)R(n, x)$, $(x)R(n, x)$, $(x)(Ey)R(n, x, y)$, $(Ex)(y)R(n, x, y)$, $\cdots$. The variables $n, x, y, \cdots$ range over nonnegative integers. The quantifiers $(E.)$ and $(.)$ express "there exists an . such that" and "for .," respectively. Choose one of the forms, and let R be a predicate (that is, propositional function) of the variables used in that form. Then the expression represents a predicate P of the variable n. As defined by Herbrand and Gödel, a general recursive predicate has a certain kind of algorithm, which, for given values of the variables, leads to a decision as to the truth or falsity in a finite number of steps. According to Church, this kind of algorithm is the most general possible. To each of the above forms, a general recursive R can be chosen, such that the resulting P is not given by any of the other forms with the same or fewer quantifiers, for any general recursive R whatsoever. This includes, for the first of the above forms, Church's result on an unsolvable problem of elementary number theory, and for the second, a part of Gödel's results on the incompleteness of formal deductive systems. The R chosen for each form is primitive recursive. (Received August 7, 1940.)

465. R. E. Langer: *On the theory of irregular differential boundary problems.* Preliminary report.

A matrix differential boundary problem $Y'=(\lambda R+B)Y$, $W_aY(a)+W_bY(b)=0$, is familiarly classifiable as regular, mildly irregular, or highly irregular, according as the characteristic equation possesses or does not possess certain structural peculiarities. In any event, if an infinity of characteristic values exists, an "arbitrary" function may be formally associated with an "expansion" in characteristic solutions of the problem. Under appropriate conditions this expansion is known to be convergent or summable to the given function if the problem is respectively regular or mildly irregular. There is, however, little in the way of general theory for the highly irregular cases. This paper seeks to obtain such theory by the study of a boundary problem which depends upon an additional parameter ν, in such a way that it reduces to the given (irregular) problem for some specific value of ν, while it remains regular for adjacent values. It is shown that in a specifiable sense the expansions associated with irregular cases of certain categories may be regarded as summable to the given function. (Received August 2, 1940.)

466. W. G. Madow: *Contributions to the theory of the representative method of sampling.*

The theory of representative sampling may be regarded as a dual process, the first consisting in sampling different random variables and the second in repeating several times the experiments associated with each of the different random variables. It follows that while the theory of sampling from finite populations without replacement may be required for the first process, the second leads directly into the theory of sampling from infinite populations. There is, however, one difference. The usual theory is concerned with the evaluation of fiducial or confidence limits for, say, the mean of a sample of N when n, $(N \geqq n)$, of the values are known. It is thus possible to use the usual theories of estimation in obtaining estimates of the parameters and to allow the effects of the subsampling process to show themselves in the different values of the fiducial limits. It is shown that the limits obtained are almost identical with those obtained by the theory of sampling from a finite population. Distributions of the statistics used in these limits are derived. Besides these results, the theory is extended to that of sampling vectors and the conditions are stated under which the "best" allocation of the number in a sample among several strata is proportional to the kth roots of the generalized variance of a random vector having k components. (Received August 2, 1940.)

467. Jerzy Neyman: *Conception of equivalence in the limit of statistical tests and its application to certain forms of χ^2 and to the tests based on the λ principle.*

Denote by T_1 and T_2 two different tests of the same hypothesis H and by N the number of observations to be used to test H. Definition: if the probability of T_1 and T_2 contradicting themselves tends to zero as N is increased, then T_1 and T_2 are called equivalent in the limit. Consider an experiment which will produce one of the results $E_1, E_2, \cdots, E_s$, the probability of E_1 being p_i. This experiment is to be repeated n times and n_i will denote the number of occurrences of E_i. Denote by H the hypothesis that $p_i = f_i(\theta_1, \theta_2, \cdots, \theta_h)$, where f_i is a function of h parameters θ_j, with continuous derivatives of second order such that $\partial(f_1, f_2, \cdots, f_h)/\partial(\theta_1, \theta_2, \cdots, \theta_h) \neq 0$. Suppose that H is tested against a set of alternatives ascribing to the p's any non-negative values, such that $\sum_i p_i = 1$. Under those conditions each of the following χ^2 tests is equivalent in the limit to the test based on the λ principle: (1) test T_1 consists in the rule of rejecting H whenever $\sum_i (n_i - np_i')^2/np_i' > \chi^2_\alpha$, where p_i' denotes the maximum likelihood estimate of p_i; (2) test T_2 consists in rejecting H whenever

$\sum_i(n_i-n\bar{p}_i)^2/n\bar{p}_i>\chi^2_\alpha$, where $\bar{p}_i$ denotes the value of p_i minimizing $\sum_i(n_i-np_i)^2/np_i$; (3) test T_3 consists in rejecting H whenever $\sum_i(n_i-np_i'')^2/n_i>\chi^2_\alpha$, where p_i'' denotes the value of p_i minimizing $\sum_i(n_i-np_i)^2/n_i$. (Received September 11, 1940.)

468. Isaac Opatowski: *On the motion of an electric particle.*

The motion of an electrically charged point mass under the action of a magnetic force $\mathfrak{H}=-\text{grad } U(P)$ and an electric force $\mathfrak{E}=-\text{grad } V(P)$ is studied in the case in which U and V are harmonic functions satisfying the conditions: (1) grad $W(P)=\mathfrak{u}\times\mathfrak{H}$, $C=\mathfrak{u}\cdot\mathfrak{E}$ where W is a certain function (generalization of the ordinary stream function), C is a scalar constant, $\mathfrak{u}=\mathfrak{a}+\mathfrak{b}\times\mathfrak{r}$, where $\mathfrak{a}$, $\mathfrak{b}$ are two constant vectors and $\mathfrak{r}$ is the radius vector. The most important fields of mathematical physics, that is, general plane fields, all fields symmetric about an axis of revolution and many others are included in (1). The integral of momentum is explicitly given in terms of W and C. Regions of space are determined within which, for a given energy and a given constant of momentum integral, no motion is possible (generalization of Stoermer's "forbidden regions"). Equations of certain trajectories are explicitly given. The paper deals with both constant and relativistic mass. (Received August 2, 1940.)

469. Gordon Pall: *On the arithmetic of ternary quadratic forms.*

The fundamentals of the arithmetic of ternary quadratics is presented with modifications of the usual invariants, leading to very considerable simplifications. Among the topics treated are: order invariants, form-residues, generic characters, five equivalent definitions of genus, the number of existing genera, simultaneous representation, simultaneous characters, the numbers represented by a genus, zero and universal forms, rational equivalence, reduction formulas for the representations of pn or p^2n, and so on. (Received August 5, 1940.)

470. Everett Pitcher: *The cap heights of a sum function on a product space.*

This paper contains a solution under some restrictions of a problem in critical point theory arising in the work of M. Morse and C. Tompkins (see Annals of Mathematics, (2), vol. 38 (1937), pp. 386–449 and vol. 40 (1939), pp. 443–472, and forthcoming papers) on the problem of Plateau. M and N denote metric spaces and $f(x)$ and $g(y)$ positive functions defined upon them. It is assumed that (i) the sets $f\leqq c<\infty$ are compact, and (ii) the cap heights of f form an increasing sequence, finite or with $+\infty$ as limit. It is assumed that g satisfies (i) and (ii) also. On $M\times N$, $h[x\times y]=f(x)+g(y)$ and satisfies (i) and (ii). Vietoris topology with coefficients from a field is used. An arbitrary maximal group of k-caps of h at height e is shown to be isomorphic to a direct sum of homology groups $H^k(c, d)$. The summation extends over all pairs c, d of cap heights of f and g respectively with sum e. The group $H^k(c, d)$ is the homology group of classes of k-cycles on the relative product space $[f\leqq c]\times[g\leqq d]d-\text{mod }[f\leqq c]\times[g<d]\cup[f<c]\times[g\leqq d]$. Some consideration is given to homology groups of a product space. (Received August 6, 1940.)

471. Hillel Poritsky and H. D. Snively: *Doubly periodic networks.*

A solution is obtained for the doubly infinite number of linear equations of a doubly periodic network. Expressions are obtained for the self and mutual impedances as double integrals which are then reduced to elliptic integrals (of all three kinds).

These impedances are also evaluated by direct numerical methods of successive approximation. (Received August 21, 1940.)

472. J. F. Randolph: *Some properties of sets of the Cantor type.*

Given $a_1 \geqq a_2 \geqq \cdots > 0$ where $\sum_n a_n = 1$, let S consist of the numbers x for which there exists a sequence $\{\epsilon_n\}$, where each ϵ_n is 0 or 1, such that $x = \sum_n \epsilon_n a_n$. (If $a_n = 2/3^n$ then S is the Cantor set.) Theorem: $a_n \geqq 2\sum_{i=n+1}^{\infty} a_i$ is a necessary and sufficient condition that corresponding to $0 \leqq z \leqq 2$ there exists an $x \varepsilon S$ and $y \varepsilon S$ such that $x + y = z$. Also some results are given on the Carathéodory and Gillespie linear measures of cartesian products of sets of the Cantor type. (Received August 6, 1940.)

473. W. T. Reid: *A new class of self-adjoint boundary value problems.*

This paper treats a differential system consisting of the vector differential equation $\mathcal{L}[y] \equiv y' - A(x)y = \lambda B(x)y$ and two-point boundary conditions $s[y] \equiv My(a) + Ny(b) = 0$ which is self-adjoint under the nonsingular transformation $z = T(x)y$ and satisfies the further conditions: (1) the matrix $S(x) = \tilde{T}(x)B(x)$ is symmetric on ab; (2) $H[y] = \int_a^b \tilde{y}\tilde{T}\mathcal{L}[y]dx > 0$ for all vectors y satisfying $s[y] = 0$, $B(x)y(x) \not\equiv 0$, and for which there exists a corresponding vector $g(x)$ such that $\mathcal{L}[y] = Bg$ on ab; (3) $y \equiv 0$ is the only vector satisfying on ab the conditions $\mathcal{L}[y] = 0$, $By \equiv 0$, $s[y] = 0$. The chief distinction between these conditions and those defining a definitely self-adjoint boundary problem (Bliss, Transactions of this Society, vol. 44 (1938), pp. 413–428) is that the above hypothesis (2) on $H[y]$ replaces a corresponding definiteness assumption on the matrix $S(x)$. Such systems are shown to possess fundamental properties similar to those previously established for definitely self-adjoint problems. In particular, certain important types of boundary problems associated with the calculus of variations which are not definitely self-adjoint do belong to this new class of problems. (Received August 6, 1940.)

474. I. J. Schoenberg: *On positive definite functions on spheres.*

The positive definite functions on m-dimensional spheres and on spheres in Hilbert space are explicitly determined. In ordinary terminology the results are as follows: I. Let $f(x)$ be real and continuous in the interval $-1 \leqq x \leqq 1$. Let $A = \|a_{ik}\|$ denote an arbitrary n-rowed symmetric matrix of a positive quadratic form whose rank does not exceed the fixed number $m+1$ $(m \geqq 1)$. Moreover $a_{11} = a_{22} = \cdots = a_{nn} = 1$. The quadratic form corresponding to the transformed matrix $f(A) = \|f(a_{ik})\|$ is also always positive if and only if $f(x)$ admits throughout $[-1, 1]$ an expansion in ultraspherical polynomials $f(x) = \sum_{\nu=0}^{\infty} p_\nu P_\nu^{((m-1)/2)}(x)$ with nonnegative coefficients $p_\nu \geqq 0$. II. The assumptions on $f(x)$ and the matrix A are as above except that there is now no restriction whatever on the rank of A. The quadratic form of $f(A) = \|f(a_{ik})\|$ is also always positive if and only if $f(x) = p_0 + p_1 x + p_2 x^2 + \cdots$, $(-1 \leqq x \leqq 1;\ p_\nu \geqq 0)$. (See Pólya and Szegö, vol. 2, p. 107, where the sufficiency of the power series expansion for $f(x)$ is stated as Problem 37.) The second theorem is derived from the first by letting $m \rightarrow \infty$. This passage to the limit is carried through by means of the following new estimate: $|P_n^{(\lambda)}(\cos\theta)| \leqq P_n^{(\lambda)}(1)\{((1+\cos^2\theta)/2)^{n/2} + (4/n)\cdot(1+\cos^2\theta)/\sin(2\theta)\}$ if $0 < \theta < \pi/2$, $\lambda \geqq 5$, and $n = 1, 2, 3, \cdots$. (Received August 5, 1940.)

475. I. M. Sheffer: *On the singularities of functions under certain linear transformations.*

The linear transformations under consideration are all of the form $L[y(x)]$

$\equiv\sum_0^\infty L_n(x)y^{(n)}(x)$. When $L[y(x)]=F(x)$, the relation between the singularities of $F(x)$ and those of $y(x)$ is examined. The theorems obtained are analogous to the well known theorems of Hadamard and Hurwitz on the singularities of power series compounded from given power series; indeed, these latter theorems are but particular instances of the results of the present work. (Received August 6, 1940.)

476. D. C. Spencer: *On an inequality of Grunsky.*

Let W be a Riemann domain which, for simplicity, is assumed to be the map of $|z|<1$ by a function $f(z)$ regular in $|z|\leqq 1$, and let $n(w)$ be the number of times the point w is covered by W. Suppose that $n(w)\leqq n(0)$. Then Grunsky (Dissertation, Schriften des mathematischen Seminars der Universität Berlin, vol. 1 (1932), pp. 113–118) proves the following inequality (a case of the theorem of the geometric and arithmetic means in which the weight function is not necessarily of constant sign): $\exp\{(1/2\pi n(0))\int_B \lg R^2 d\Phi\}\leqq(1/2\pi n(0))\int_B R^2 d\Phi$, where B is the boundary of W. In this note a new and simple proof of the inequality is given for more general domains W which are only mean $n(0)$-valent (that is, under conditions less restrictive than $n(w)\leqq n(0)$). The inequality is applied to obtain a more precise version of a theorem of Bermant (Comptes Rendus de l'Académie des Sciences, Paris, vol. 207 (1938), pp. 31–33). (Received August 6, 1940.)

477. Gabor Szegö: *Power series with multiply monotonic sequences of coefficients.*

A sequence $\{a_0, a_1, \cdots\}$ is called monotonic of order k if the differences $\Delta^{(v)}a_n$ defined by $\Delta^{(0)}a_n=a_n$, $\Delta^{(1)}a_n=a_n-a_{n+1}$, $\Delta^{(v)}=\Delta^{(1)}\Delta^{(v-1)}$ are non-negative for $0\leqq v\leqq k$, $n\geqq 0$. Let $a_n\not\equiv 0$. Fejér proved (Transactions of this Society, vol. 39 (1936), p. 57) that $\sum a_n z^n$ is regular and univalent for $|z|<1$ provided $\{a_n\}$ is monotonic of order $k=4$. He also showed that this is not true for $k=1$. In the present paper it is proved that the assertion in question remains true for $k=3$, but not for $k=2$. The latter fact has been established by a different method by Mr. S. Szidon (Acta Szeged, 1940). (Received September 11, 1940.)

478. W. J. Trjitzinsky: *Properties of growth for solutions of differential equations of dynamical type.*

In the background of this work are, in part, certain memoirs of A. Liapounoff, P. Bohl, E. Cotton, and O. Perron. The developments for linear systems are based on characteristic numbers and product integration. The nonlinear systems are investigated, in part, with the aid of the theory of linear systems. In the study of the nonlinear problem, use is made of characteristic numbers and of successive approximations. The latter method is associated with some of the present author's earlier work in the field of differential equations. (Received August 5, 1940.)

479. Abraham Wald: *Asymptotically shortest confidence intervals.*

Let $f(x, \theta)$ be the probability density function of a variate x involving an unknown parameter θ. Let $x_1, x_2, \cdots, x_n$ be n independent observations on x and let $C_n(\theta)$ be a positive function of θ such that the probability that $F(x, \theta)\equiv|n^{-1/2}\partial\sum_{\alpha=1}^n \log f(x_\alpha, \theta)/\partial\theta|\leqq C_n(\theta)$ is a constant β when θ is the true value of the parameter. Denote by $\underline{\theta}(x_1, \cdots, x_n)$ the root in θ of the equation $F(x, \theta)=C_n(\theta)$ and by $\bar{\theta}(x_1, \cdots, x_n)$ the root of $F(x, \theta)=-C_n(\theta)$. Under some weak assumptions on $f(x, \theta)$ the limiting interval $\delta_n(x_1, \cdots, x_n) = [\underline{\theta}(x_1, \cdots, x_n), \bar{\theta}(x_1, \cdots, x_n)]$ for $n \to \infty$ is a shortest unbiased confidence interval (for the definition of a shortest unbiased confidence interval, see J. Neyman, Philosophical Transactions of the Royal Society of London, vol. 236 (1937), pp. 333–380) of θ corresponding to the confidence coefficient β. This confidence interval is identical with that given by S. S. Wilks (Annals of Mathematical Statistics, vol. 9 (1938)). Wilks has shown that $\delta_n(x_1, \cdots, x_n)$ is asymptotically shortest in the average compared with all confidence intervals computed on the basis of statistics belonging to a certain class C. In the present paper it has been proved that the confidence interval in question is asymptotically shortest compared with any arbitrary unbiased confidence interval, without any restriction to a certain class of functions. (Received August 5, 1940.)

480. Norbert Wiener and Aurel Wintner: *Discrete chaos.*

The ordinary method of introducing the Lebesque integration needed in statistical mechanics in terms of a probability density referred to a product space works in systems of a finite number of degrees of freedom, but breaks down in the case of an infinity of degrees of freedom. An alternative method is developed in this latter case, depending on an infinity of probability densities of finite sets of particles. (Received September 11, 1940.)

481. S. S. Wilks: *On the problem of two samples from normal populations with unequal variances.*

Suppose O_{n_1} and O_{n_2} are samples of n_1 and n_2 elements from normal populations Π_1 and Π_2 respectively. Let a_1, σ_1^2 and a_2, σ_2^2 be the means and variances of Π_1 and Π_2 and let O_{n_1} and O_{n_2} have means $\bar{x}_1$ and $\bar{x}_2$ and variances s_1^2 and s_2^2 (unbiased estimates of σ_1^2, σ_2^2) respectively. It is shown that there exists no function (Borel measurable) of $\bar{x}_1$, $\bar{x}_2$, s_1^2, s_2^2, $a_1 - a_2$ independent of σ_1^2 and σ_2^2, having its probability law independent of the four population parameters. It is therefore impossible to obtain exact confidence limits for $a_1 - a_2$ corresponding to a given confidence coefficient. Functions of the four parameters and four statistics are devised from which one can set up confidence limits for $a_1 - a_2$ with associated confidence coefficient inequalities. (Received August 2, 1940.)

482. W. L. G. Williams: *The application of hyperbolic complex numbers to the geometry of the triangle.*

A complex number $a+bj$, in which a and b are real numbers and $j^2=1$, is called a hyperbolic complex number. If such a number is designated by z and is represented as a two-dimensional vector, the vector zj is said to be pseudo-perpendicular to z. When the concept pseudo-perpendicular takes the place of perpendicular, new points, related to a triangle, are found, analogous to orthocentre, circumcentre and nine-point centre. These points lie on a straight line with the centroid of the triangle. There exist rectangular hyperbolas analogous to the circumcircle and nine-point circle. There is an analogue of Feuerbach's theorem. (Received August 6, 1940.)

483. Antoni Zygmund: *Power series of the class H^λ on the circle of convergence.*

If the function $f(z) = \sum c_n z^n$ belongs to the class H^λ, where $0 < \lambda < 1$, then the series $\sum c_n e^{in\theta}$ is summable $(C, 1/\lambda - 1)$ for almost every θ. (Received August 19, 1940.)

484. A. T. Brauer: *On a problem of partitions.*

Let $a_1, a_2, \cdots, a_k$ be relatively prime positive integers. The purpose of this paper is to determine bounds $F(a_1, a_2, \cdots, a_k)$ such that the Diophantine equation $a_1x_1+a_2x_2+\cdots+a_kx_k=n$ always has solutions in positive integers $x_1, x_2, \cdots, x_k$ for $n>F(a_1, a_2, \cdots, a_k)$. (Received September 27, 1940.)

485. A. B. Brown: *On the number of independent parameters.*

Let $f(x_1, \cdots, x_n, \alpha_1, \cdots, \alpha_m)$ be a function of class $C^{(q)}$ (q specified in the paper). To form matrix (M), take $\partial f/\partial\alpha_1, \cdots, \partial f/\partial\alpha_m$ as first row. As further trial rows, take the derivatives to $\alpha_1, \alpha_2, \cdots, \alpha_m$ of derivatives of f to the x's, in the following order: $\partial f/\partial x_n$, $\partial f/\partial x_{n-1}, \cdots, \partial f/\partial x_1$, $\partial f/\partial x_n^2, \cdots$. If the set of derivatives to $\alpha_1, \cdots, \alpha_m$ increases the rank of the matrix, the derivative, say $\partial z/\partial x_n$, is *accepted*, otherwise *rejected*. The algorithm continues until the accepted derivatives are the only derivatives of z which are not rejected derivatives or derivatives of rejected derivatives. If k is the number of rows of the resulting matrix, then *all functions f are given by varying k of the parameters, keeping the other $n-k$ constant*; *furthermore, no function is given twice in this way.* Certain "singular points," forming a nowhere dense closed set, must be avoided. It is proved that not all functions can be obtained by varying less than k parameters. Further, given $g(x_1, \cdots, x_n, \beta_1, \cdots, \beta_t)$ where $f(x, \alpha)$ and $g(x, \beta)$ are the same set of functions of (x), but nothing is assumed as to relationship between (α) and (β), the same number k of parameters is needed for $g(x, \beta)$. (Received September 25, 1940.)

486. M. M. Day: *Some more uniformly convex spaces.*

For a sequence of Banach spaces $[B_i]$, $i=1, 2, \cdots$, define $B=\prod^p\{B_i\}$ to be the space of sequences $b=\{b_i\}$ with $b_i\in B_i$ and $\|b\|=(\sum\|b_i\|^p)^{1/p}<\infty$. Complementing an earlier result of this writer (abstract 46-9-396) it is shown that $\prod^p\{B_i\}$ is uniformly convex if $B_i=l^{p_i}$ or L^{p_i} and if $1<m\leqq M<\infty$ exist with $m\leqq p_i\leqq M$ for all i. A conjecture of Boas (this Bulletin, vol. 46 (1940), p. 304) that the spaces $\prod^p\{B_i\}$ with $B_i=B_0$ for all i, as well as the spaces $L^p(B_0)$ (consisting of all Bochner integrable functions f on, say, (0, 1) with values in B_0 and $\|f\|=(\int_0^1|f(t)|^pdt)^{1/p}<\infty$), are uniformly convex if B_0 is shown to be true. A sequence of spaces $\{B_i\}$ is said to have a common modulus of convexity if for each ϵ, $0<\epsilon\leqq 2$, there is a $\delta(\epsilon)>0$ such that for each i and each pair of points $b_i, b_i'\in B_i$ with $\|b_i\|=\|b_i'\|=1$ and $\|b_i-b_i'\|>\epsilon$, the relation $\|b_i+b_i'\|<2(-\delta(\epsilon))$ holds. The results mentioned above are corollaries of the following theorem: $\prod^p\{B_i\}$ is uniformly convex if and only if the B_i have a common modulus of convexity. (Received September 21, 1940.)

487. Aaron Fialkow: *The conformal theory of a hypersurface.*

For a hypersurface V_{n-1} in any Riemann space V_n $(n>2)$, the author shows that it is possible to define three quadratic differential forms which remain unchanged under any conformal mapping of V_n, not necessarily on itself. These forms are the *conformal fundamental forms* of the hypersurface V_{n-1}. If $n>3$, the coefficients of the third conformal fundamental form may be expressed in terms of the coefficients of the first and second conformal fundamental forms. If V_{n-1} and $\bar{V}_{n-1}$ are hypersurfaces of V_n and $\bar{V}_n$ respectively and $V_n\leftrightarrow\bar{V}_n$, $V_{n-1}\leftrightarrow\bar{V}_{n-1}$ by a conformal map, then the conformal fundamental forms of V_{n-1} and $\bar{V}_{n-1}$ are equal. Conversely, if V_n and $\bar{V}_n$ are conformally euclidean spaces and the conformal fundamental forms of V_{n-1} and $\bar{V}_{n-1}$ are equal, then a conformal transformation exists so that $V_n\leftrightarrow\bar{V}_n$, V_{n-1}

$\leftrightarrow \bar{V}_{i-1}$. In any V_n, a hypersurface exists whose conformal fundamental forms are any preassigned quadratic differential forms whose coefficients satisfy certain partial differential equations analogous to the classical Gauss-Codazzi equations. The Weyl conformal curvature tensor of V_n plays a role analogous to that of the Riemann curvature tensor in classical differential geometry. The case $n=3$ occupies a special position in this theory. (Received September 23, 1940.)

488. G. E. Forsythe: *On Riesz summability methods of order r, for $\Re(r)<0$.* Preliminary report.

Let A_r and B_r represent M. Riesz's (discrete *resp.* continuous) summability methods for sequences, as defined by Agnew (Transactions of this Society, vol. 35 (1933), pp. 532–548). It is known that A_r and B_r and Cesàro C_r are equivalent for $-1<r\leqq 1$, while A_r and B_r are not equivalent for the real part of r, $\Re(r)<-1$, nor for certain values of $r>1$. By aid of Theorem 5.1 (op. cit.) the present paper obtains a criterion for the equivalence of A_r and B_r for $\Re(r)<0$ in terms of the reciprocal of $\phi_r(x)=\sum_1^\infty n^r x^n$, and applies it to prove that A_{-1+ih} and B_{-1+ih} $(-\infty<h<\infty)$ are equivalent if and only if $h=0$. (Received September 25, 1940.)

489. Saul Gorn: *Homomorphisms and modular functionals.*

Using CA to represent the complements of elements of A, an ideal in a complemented modular L is called a C-ideal if $C\mathfrak{A}$ is an ideal; $C\mathfrak{A}$ is a π-ideal if $\mathfrak{A}$ is a σ-ideal and dually. If $\mathfrak{A}=CC\mathfrak{A}$, $\mathfrak{A}$ is called neutral (identical with Garrett Birkhoff's definition). An $\mathfrak{A}$ fulfilling both conditions is called C-neutral. The general homomorphism theorem for complemented modular lattices is given by the C-neutral ideals. Letting $\mathfrak{A}'$ be the elements orthogonal to all elements of $\mathfrak{A}$, a C-neutral ideal is normal if and only if it is normal in Stone's sense: $\mathfrak{A}''=\mathfrak{A}$. The homomorphism $L\rightarrow L/\mathfrak{A}$ takes normal ideals containing or equal to $\mathfrak{A}$ into normal ideals; conversely if $\mathfrak{A}$ is normal. L is called ideally irreducible if it contains no non-trivial C-neutral normal ideal. $L/\mathfrak{A}$ is ideally irreducible if and only if $\mathfrak{A}$ is a maximal (or prime for π-ideals) C-neutral normal ideal. Using the Wilcox-Smiley continuity conditions, a quasi-norm is uniquely determined by its generating normal C-neutral $\mathfrak{A}$ if and only if $L/\mathfrak{A}$ is ideally irreducible (that is, $\mathfrak{A}$ maximal if non-trivial). This generalizes results mentioned in abstract 45-1-16. (Received September 12, 1940.)

490. N. A. Hall: *The solution of the quadrinomial equation in infinite series by the method of iteration.*

The roots of the quadrinomial equation $z^{m+n+k}-z^{m+n}+bz^m+a=0$ may be expressed as double power series in the two complex variables a and b. These solutions have been previously developed by use of the Lagrange expansion and also by use of differential resolvents. In this paper, these results are reproduced and extended by the direct method of solution of the equation by iteration as previously applied by the author to the trinomial equation (this Bulletin, vol. 44 (1938), p. 337). The solutions are exhibited as generalized hypergeometric series in the two variables a and b. These series give the set of $m+n+k$ roots covering the quadrant of absolute values of a and b except for the small region covered by the branch points of the roots. The relation between these singularities and the domains of convergence for the double series is shown. (Received September 27, 1940.)

491. Einar Hille: *A class of differential operators of infinite order.*

The author studies the differential operator $G(\delta_z)$ where $G(w)$ is an entire function and $\delta_z = z^2 - d^2/dz^2$ is the operator of the Hermite-Weber equation. Necessary and sufficient conditions are found in order that $G(\delta_z)$ shall apply to various classes of analytic functions, in particular entire functions. The conditions differ in some respects fundamentally from corresponding conditions for the operator $G(d/dz)$. Relations are found between the orders and types of $G(w)$, $f(z)$, and $G(\delta_z)\cdot f(z)$. The equation $G(\delta_z)\cdot W = F(z)$ is studied with the aid of summable Fourier-Hermite series. If $F(z) \sim \sum F_n h_n(z)$, then formally $W(z) \sim \sum W_n h_n(z)$, where $G(2n+1)W_n = F_n$. Sufficient conditions are found under which this formal procedure leads to an actual solution and a uniqueness theorem is proved. Some examples are discussed which show that if the operator δ_z is replaced by other second order differential operators D_z then the applicability of $G(D_z)$ to entire functions may be governed by completely different conditions. (Received August 30, 1940.)

492. Loo-Keng Hua: *Some "Anzahl" theorems in the theory of groups of prime-power order.*

A group G of order p^n (p being a prime) is said to be of rank δ if the maximum of the order of elements of G is equal to $p^n - \delta$. The author selects the following theorem to be announced: If G is a group of order p^n ($p \geqq 3$, $u \geqq 2\alpha$) of rank α, then (i) G contains one and only one subgroup of order p^n ($\alpha \leqq m \leqq n$) of rank α; (ii) G contains p^α cyclic subgroups of order p^m ($\alpha < m < n - \alpha + 1$) and (iii) G contains $p^{m+\alpha}$ ($\alpha \leqq m \leqq n - \alpha$) elements which satisfy $x^{p^m} = 1$. The second and the third statement solve the "Anzahl" theorems of Miller's and Kalakoff's type. (Received August 31, 1940.)

493. Loo-Keng Hua: *The lattice points in a circle.*

Let $R(x)$ denote the number of lattice points in the circle $m^2 + n^2 = x$. The object of the paper is to prove that $R(x) = \pi x + O(x^{\alpha+\epsilon})$, $\alpha = 13/40$, which is better than Titchmarsh's result (Proceedings of the London Mathematical Society, (2), vol. 38 (1933), pp. 96–115) with an error term $O(x^{15/46+\epsilon})$. Notice that Vinogradow's proof of the error term $O(x^{17/53+\epsilon})$ is incorrect (Bulletin de l'Académie des Sciences de l'URSS, vol. 7 (1932), pp. 313–336). (Received August 31, 1940.)

494. W. H. Ingram: *A generalization of Erhard Schmidt's solution of the nonhomogeneous integral equation.* Preliminary report.

It is first proved that the Hilbert transform of an arbitrary step-wise continuous vector matrix has, in terms of the characteristic solutions of the equation $\phi_r(x) = \lambda_r \cdot \int_b^a \mathfrak{K}(x, \alpha)\phi_r(\alpha)d\alpha$, an absolutely and uniformly convergent development which represents the transform in the sense of least squares when the scalar coefficients are computed by means of the associated characteristic functions ψ_r. As in abstract 46-5-282, $\mathfrak{K}$ is a non-symmetric square matrix, and ϕ_r and ψ_r, respectively, column and line matrices which can be biorthonormalized. In this sense of equality and on the basis of this expansion theorem, the generalization of Schmidt's result previously announced is independently derived. (Received August 27, 1940.)

495. L. H. Loomis: *The decomposition of meromorphic functions into rational functions of univalent functions.*

The general question discussed is this: given $f(z)$ meromorphic in $|z| < 1$, when does there exist a decomposition $f(z) \equiv f_2(f_1(z))$ where $\zeta = f_1(z)$ is univalent in $|z| < 1$, and $f_2(\zeta)$ is a rational function? A simple but typical result is that if $f(z)$ is meromorphic in $|z| \leqq 1$, then the decomposition is possible. Necessary and sufficient conditions are derived for the general case. Stronger theorems are proved for situations in which $\zeta = f_1(x)$ is the mapping function of a simply-connected plane region which is not of the most general type, for instance, a Jordan region. Analogous theorems are developed for $f(z)$ regular and bounded in $|z| < 1$ and $f_2(\zeta)$ restricted to special types of rational functions, say polynomials. In proof, the author considers the possibility of imbedding the Riemann surface which is the map of $|z| < 1$ under $w = f(z)$ in the Riemann surface of the inverse of a rational function. (Received September 23, 1940.)

496. P. T. Maker: *A topological characterization of monotone functions.* Preliminary report.

Let $f(s)$, a function from S to X, both topological spaces, be in $M(S, X)$ if the complete inverse image of every connected set in X is connected. It is shown that $M(R_1, R_1)$ is the class of monotone functions, and that for other spaces M includes the functions "monotone" by previous generalizations. Conditions on S and X, in order that $f(s)$ have certain continuity properties, or that M be complete or compact, are determined. (Received September 28, 1940.)

497. E. R. Ott: *A locus determined by an algebraic correspondence.*

Let P and Q be two variable vertices of a hexagon inscribed in a conic and let the other four vertices be fixed. If P and Q have parameters $t' = F(t)$ and $t' = G(t)$, where $F(t)$ and $G(t)$ are functions which establish an (m, n) algebraic correspondence between P and Q, then as t varies the Pascal line of the hexagon envelopes a curve of class $(m+n)$ and of order $2(m+n-1)$. Each of the two fixed sides of the hexagon which are adjacent to the variable side is a multiple tangent of the locus. All of the Pluecker characteristics of the curve are obtained and are independent of the order of the vertices of the hexagon. (Received September 14, 1940.)

498. Francisco Perez: *A generalization of the theory of invariant factors and similar matrices.*

E. H. Moore's general identity matrix $\delta_{\mathfrak{M}}$ (E. H. Moore, *General Analysis*, Part I) for a right linear space $\mathfrak{M}$ of vectors on a finite range $\mathfrak{P}$ to $\mathfrak{A}$ is shown to exist even when $\mathfrak{A}$ is a number system of *type B*, provided $\mathfrak{M}$ has the property of being *perfect*. In a perfect space $\mathfrak{M}$ of rank r, to every matrix ϕ of type $\mathfrak{M}\overline{\mathfrak{M}}$ and of rank r corresponds uniquely a matrix ϕ^{-1} of type $\mathfrak{M}\overline{\mathfrak{M}}$ such that $S\phi^{-1}\phi = S\phi\phi^{-1} = \delta_{\mathfrak{M}}$. When multiplication is commutative, it is shown (without using determinants) that every matrix κ of type $\mathfrak{M}\overline{\mathfrak{M}}$ determines uniquely a set of polynomials $P_i(z)$ $(i = 1, 2, \cdots, t)$ called the *invariant factors of* κ *in* $\mathfrak{M}$. The sum of the degrees of these polynomials is equal to r, and P_{i+1} divides P_i. It is shown that $P_1(\kappa) = O(\mathfrak{P}\mathfrak{P})$, where $\kappa^0 \equiv O(\mathfrak{P}\mathfrak{P})$. Two matrices κ_1 and κ_2 of type $\mathfrak{M}\overline{\mathfrak{M}}$ are said to be *similar in* $\mathfrak{M}$ if there exists a matrix ϕ of type $\mathfrak{M}\overline{\mathfrak{M}}$ and of rank r such that $\kappa_2 = SS\phi^{-1}\kappa_1\phi$. Two matrices of type $\mathfrak{M}\overline{\mathfrak{M}}$ are similar in $\mathfrak{M}$ if and only if they have the same invariant factors in $\mathfrak{M}$. (Received September 14, 1940.)

499. L. B. Robinson: *A functional equation with a singular line.*

The equation (I) $u'(x) = a(x)u(x^2)$ can be transformed into (II) $w'(z) = e^z a(e^z) w(2z)$ which in general has a solution with $z = 0$ as singular point. The exceptional case is given by the vanishing of a determinant. In general the circumference of the unit circle is a singular line of $u(x)$. The necessary and sufficient condition for the failure of this rule has been found. (Received September 14, 1940.)

500. L. B. Robinson: *A functional equation with negative exponent.*

Consider the system $u_x'(x)=(\lambda/(1+x^2))[P_n(x)/Q_n(x)]u(x^{-2})$, $u_x'(x^{-1})=(-\lambda/(1+x^2))[P_n(x^{-1})/Q_n(x^{-1})]u(x^2)$. The solution of this system is $u(x)\equiv u_0\{1+r\lambda\phi_1(x)+\lambda^2\phi_2(x)+r\lambda^3\phi_3(x)+\cdots\}$, $u(x^{-1})\equiv u_0\{r+\lambda\psi_1(x)+r\lambda^2\psi_2(x)+\lambda^3\psi_3(x)+\cdots\}$ where $\phi_s(x)\equiv\int_0^x(1/(1+\xi^2))[P_n(\xi)/Q_n(\xi)]\psi_{s-1}(\xi^{-2})d\xi$, $\psi_s(x)\equiv\int_0^x(-1/(1+\xi^2))[P_n(\xi^{-1})/Q_n(\xi^{-1})]\cdot\phi_{s-1}(x^2)d\xi$. If the value of r is rightly determined, the two series $u(x)$ and $u(x^{-1})$ converge and are consistent. (Received September 14, 1940.)

501. Jenny E. Rosenthal: *Generating functions and properties of certain orthogonal polynomials.* II.

The nonhomogeneous differential equation satisfied by generating functions for orthogonal polynomials reduces in the simplest possible case to an algebraic equation. Its solution is the generating function of a certain type of Szegö-Bernstein polynomials. A method is given for obtaining the weight factor from the generating function and for finding some additional properties of these polynomials. The second order differential equation satisfied by the polynomials is derived and is shown to reduce in a special case to an equation which was obtained by Shohat. (Received September 27, 1940.)

502. Peter Scherk: *On real closed curves of the order $n+1$ in the projective n-space.* Preliminary report.

The author discusses closed curves K^{n+1} in the real projective n-space which have osculating spaces of all the dimensions less than n and which have the real order $n+1$; that is, the maximum number of points of intersection of a K^{n+1} with an $(n-1)$-space shall be $n+1$. These curves are a generalization of the algebraic curves of order $n+1$ with one real branch. Analogous to the algebraic case, singular points can be defined and provided with multiplicities. The sum of the multiplicities of the singular points of a K^{n+1} is less than or equal to $n+1$ and congruent to $n+1$ (mod 2). The types for which this sum equals $n+1$ are characterized. Furthermore, the pairs of curves in the projective n-space are described which have no more than $n+1$ points with any $(n-1)$-space in common. These pairs generalize the algebraic curves of order $n+1$ with two real branches. Some special qualities of the K^{n+1} are indicated. The main tool of the discussion is the continuous transformation of the K^{n+1} into itself which correlates to each point of the K^{n+1} the point at which its osculating $(n-1)$-space intersects the K^{n+1} again. (Received September 27, 1940.)

503. W. T. Scott and H. S. Wall: *A geometrical method in the theory of continued fractions.*

In this paper the authors consider continued fractions of the form $1/1+a_2/1+a_3/1+a_4/1+\cdots$, where the a_n's are complex numbers. By regarding the continued fraction as a succession of linear fractional transformations they determine conditions under which the nth approximant A_n/B_n of the continued fraction will lie in a region V of the complex plane whenever the a_n's lie in a region U. Under these conditions any convergent continued fraction whose elements lie in U must have a value in V. As a supplement to their recent "parabola" theorem (Transactions of this Society, vol. 47 (1940), p. 166) they show that if the a_n's lie in the parabola $|z|-R(z)\leqq\frac{1}{2}$ every approximant A_n/B_n lies in the circle $|z|^2-2R(z)\leqq 0$, and that this is the "best" circular region having the above property. (Received August 19, 1940.)

504. W. T. Scott and H. S. Wall: *Linear manifolds of Hausdorff means.*

A set M of moment sequences is a manifold if when $\{c_n\}$, $\{d_n\}$ are in M, then $\{c_n+d_n\}$ and $\{rc_n\}$ (r real) are in M. A manifold M is regular if when $\{c_n\}\in M$, $c_0\neq 0$ then $\{c_n/c_0\}$ is a regular Hausdorff sequence. The authors determine conditions under which a sequence $\{\beta_n(u)\}$ of real functions on the interval (0, 1) constitutes a basis for a manifold $M(\beta)$, in the sense that M is made up of the sequences $c_n=\int_0^1\beta_n(u)\cdot d\phi(u)$, $(n=0, 1, 2, \cdots)$, for $\phi(u)\in BV[0,1]$. They also determine conditions upon the basis such that $M(\beta)$ shall be regular, and conditions such that every regular Hausdorff mean determined by the sequences of $M(\beta)$ shall include a given Hausdorff mean. The general theory is applied to a number of special examples. Manifolds exist which include any given Hausdorff mean, and, in particular, which include (C, α) but not $(C, \alpha+\epsilon)$, $(\alpha>0, \epsilon>0)$. For example, if $\beta_n(u)=(u+1)/(u+n+1)$, then $M(\beta)\supset(C, 1)$ but not $(C, 1+\epsilon)$, $(\epsilon>0)$. (Received August 19, 1940.)

505. I. E. Segal: *The space of Besicovitch almost periodic functions.*

Let B^p $(1\leqq p<\infty)$ be the space of all Besicovitch almost periodic functions of order p on the infinite interval $(-\infty, \infty)$ with the usual norm. It is shown that there is a bicompact space S and a completely additive measure μ over the field of Borel subsets of S such that B^p is isometric with the space $L_p(S, \mu)$ of all complex-valued functions which are measurable and whose pth power is summable with respect to μ; the norm is the usual one. This result follows readily from a recent theorem of Gelfand on the representation of the class of Bohr almost periodic functions. It is also connected with some recent work of Bochner and of Wecken. A similar result holds for the spaces of almost periodic functions on any locally bicompact group. (Received September 28, 1940.)

506. William Wernick: *Distributive properties of set operators.*

A point set operator a which maps a set A of a given space S uniquely into a set $aA\in S$ may have certain distributive properties defined by expressions of the form $a(Af_1B)\cdot R{:}aA\cdot f_2\cdot aB$; where f_1, f_2 are set operators $+$ or $\cdot$; and where R is a set relation, either $=$, $\subset$, or $\supset$. There are twelve such properties, but *monotonicity* and *inverse monotonicity* are included to obtain a list of fourteen "distributive" properties. These fourteen properties (α_i) and their negations $(\bar{\alpha}_i)$ form a collection of 28 properties which may be assumed or deduced for set operators. Their interrelations are investigated. The distributive character of a is determined with respect to α_i if it is known that a has property α_i $(a{:}\alpha_i)$; or that $a{:}\alpha_i$. An operator a is "completely determined" if its distributive character is determined with respect to every α_i. In view of the interrelations among the α_i, there are relatively few "completely determined" operators. A list of 25 is given, with examples. (Received September 13, 1940.)

507. William Wernick: *Functional dependence in the calculus of propositions.*

In the two-valued calculus of propositions a function F of n variables is completely determined by 2^n constants, independently 0, or 1 (call their sum S). F is independent of a particular variable x_i if no change in F results from a change in x_i alone. A function A_i is defined whose vanishing is a necessary and sufficient (NS) condition that F be independent of x_i. In terms of the A_i a function $A(F)$ is defined whose nonvanishing is a NS condition that F be really a function of all its arguments, that is, not independent of any of them. Another function B_i is defined, and it is shown that $S=2B_i$ is another NS condition that F be independent of x_i. From this are obtained very simply the following two theorems: "A sufficient condition that F be really a function of all its arguments is that S be odd." "A necessary condition that F be independent of some or all of its arguments is that S be even." (Received September 20, 1940.)

508. Hassler Whitney: *The mappings of a 3-complex into a space with vanishing fundamental group.*

Extending methods used formerly for classifying the mappings of a 3-complex into a 2-sphere S^2 (see this Bulletin, vol. 42 (1936), p. 338), S^2 is now replaced by any space S without fundamental group. The 2-dimensional homotopy group $\pi^2(S)$ is a direct sum of cyclic groups, with generators $\rho_1, \cdots, \rho_n$. There is a natural function $\theta(\rho)\in\pi^3(S)$ and a multiplication $\rho\times\rho\in\pi^3(S)$; $2\theta(\rho)=\rho\times\rho$. If $a\rho=0$, then $2a\theta(\rho)=0$. If f is a mapping of the 2-dimensional part K^2 of K into S, a 2-π^2-chain X^2 is defined (compare Duke Mathematical Journal, vol. 3 (1937), pp. 51–55); f may be extended through K^3 and K^4 if and only if $\delta X^2=0$ and $X^2\cup^* X^2\sim 0$. Here, if $\sigma^r\cup\sigma^s=\sigma^{r+s}$, then $\sum\alpha_i\rho_i\sigma^r \cup^*\sum\beta_i\rho_i\sigma^s=[\sum\alpha_i\theta(\rho_i)+\sum_{i<j}\alpha_i\beta_j(\rho_i\times\rho_j)]\sigma^{r+s}$ if no ρ_i are of finite order. Two mappings f and f' are homotopic if and only if their difference is $\sim 2(Y^1\cup^* X^2)$ for some 1-π^2-cocycle Y^1. (The factor 2 was omitted in the abstract cited above.) (Received September 28, 1940.)