ON TRANSLATIONS OF FUNCTIONS AND SETS[1]

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1. **Introduction.** It is the object of this note to prove the following theorem and two lemmas (see §3) on translations of sets which are used in the proof of the theorem.

THEOREM 1. *In order that a sequence $x_n(t)$ of complex-valued functions measurable over $-\infty<t<\infty$ may be such that, for each real sequence λ_n,*

$$\lim_{n\to\infty} x_n(t-\lambda_n) = 0 \tag{1}$$

for almost all t, it is necessary and sufficient that for each $\delta>0$

$$\sum_{n=1}^{\infty} \underset{-\infty<h<\infty}{\text{l.u.b.}} \left| E_t\{h \leqq t \leqq h+1;\ |x_n(t)| \geqq \delta\} \right| < \infty. \tag{2}$$

Necessity for Theorem 1 is established by proving the following more incisive theorem.

THEOREM 2. *If a sequence $x_n(t)$ of complex-valued functions measurable over $-\infty<t<\infty$ is such that, for each real sequence λ_n,*

$$\lim_{n\to\infty} x_n(t-\lambda_n) = 0$$

for each t in some set D of positive measure (where the set D may depend upon the sequence λ_n), then (2) holds.

Measure is that of Lebesgue, and a property such as (1) holds for almost all t if it holds for all t in the infinite interval $-\infty<t<\infty$ with the possible exception of a null set (set of measure 0). The set

$$A \equiv A(h, t, n, \delta) = E_t\{h \leqq t \leqq h+1;\ |x_n(t)| \geqq \delta\}$$

is the set of all points t such that $h\leqq t\leqq h+1$ and $|x_n(t)|\geqq\delta$; and $|A|$ denotes the measure of A. The condition (2) implies that when n is large the function $|x_n(t)|$ is less than δ for "most" values of t in each unit interval; but (2) implies no restriction whatever on $x_n(t)$ when t lies in the "exceptional" set.

The hypothesis that (1) holds for almost all t for each real *bounded* sequence λ_n does not imply (2). For example if, for each $n=1, 2, 3, \cdots$, $x_n(t)$ is a constant c_n over the interval $2^n<t<2^n+1$ and is 0 otherwise, and λ_n is a bounded sequence, then (1) holds for

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each t; but (2) fails in case c_n fails to converge to 0 as n becomes infinite.

2. **Proof of sufficiency for Theorem** 1. Let $x_n(t)$ be a sequence of measurable functions for which (2) holds, and let λ_n be a sequence of real numbers. It follows from (2) that, for each $\delta>0$,

$$(3)\qquad \sum_{n=1}^{\infty} \underset{-\infty<h<\infty}{\text{l.u.b.}} \left| E_t\{h \leqq t \leqq h+1;\ |x_n(t-\lambda_n)| \geqq \delta\}\right| < \infty .$$

Let J denote an arbitrary finite interval. Since J can be covered by a finite set of unit intervals $h\leqq t\leqq h+1$, it follows from (3) that for each $\delta>0$

$$(4)\qquad \sum_{n=1}^{\infty} \left| E_t\{t\,\varepsilon\, J;\ |x_n(t-\lambda_n)| \geqq \delta\}\right| < \infty .$$

Setting

$$(5)\qquad A_{n,p} = E_t\{t\,\varepsilon\, J;\ |x_n(t-\lambda_n)| \geqq p^{-1}\}, \qquad n, p = 1, 2, 3, \cdots,$$

we see that (4) implies existence of indices $n_1<n_2<n_3<\cdots$ such that

$$(6)\qquad \sum_{n=n_p}^{\infty} |A_{n,p}| < 2^{-p-1}, \qquad p = 1, 2, \cdots .$$

Setting

$$(7)\qquad A_r = \sum_{p=r}^{\infty} \sum_{n=n_p}^{\infty} A_{n,p}, \qquad r = 1, 2, \cdots,$$

we find

$$(8)\qquad |A_r| \leqq \sum_{p=r}^{\infty} \sum_{n=n_p}^{\infty} |A_{n,p}| < \sum_{p=r}^{\infty} 2^{-p-1} = 2^{-r}, \qquad r = 1, 2, \cdots .$$

Let

$$(9)\qquad J_r = J - A_r, \qquad r = 1, 2, \cdots .$$

If $t\,\varepsilon\, J_r$ then, when $p>r$,

$$(10)\qquad |x_n(t-\lambda_n)| < p^{-1}, \qquad n \geqq n_p,$$

so that $x_n(t-\lambda_n)$ converges to 0 over J_r. Hence $x_n(t-\lambda_n)$ converges to 0 over $J_1+J_2+\cdots$. But J_r is a subset of J having measure greater than $|J|-2^{-r}$; hence $J_1+J_2+\cdots$ is a subset of J having measure $|J|$. Therefore $x_n(t-\lambda_n)$ converges to 0 for almost all t in J.

Since J is an arbitrary finite interval, $x_n(t-\lambda_n)$ must converge to 0 for almost all t in $-\infty<t<\infty$ and sufficiency for Theorem 1 is proved.

3. **Lemmas on translations of sets.** In this section we prove two lemmas. The first states that if C and B are measurable subsets of unit intervals, then it is possible to translate B in such a way that the intersection of C and the translation of B will have measure at least $\frac{1}{2}|C|\,|B|$. The first lemma is used in proof of the second which specifies conditions under which a given sequence of sets can be translated so as to cover each point of the interval $-\infty<t<\infty$, with the exception of a null set, an infinite number of times. The close connection established in §4 between Lemma 2 and Theorem 2 shows that the combined proofs of Lemmas 1 and 2 furnish essentially a proof of Theorem 2.

If E is a set of points t in the interval $-\infty<t<\infty$ and λ is a real number, let $E(\lambda)$ denote the set of points t such that $t-\lambda \in E$; thus $E(\lambda)$ is the set obtained by translating the set E to the right λ units. Let U denote the unit interval $0\leqq t\leqq 1$.

LEMMA 1. *If C and B are measurable subsets of U, then*

$$\max_{-1\leqq\lambda\leqq 1} |CB(\lambda)| \geqq \tfrac{1}{2}|C|\,|B|. \tag{11}$$

Let $\phi(t)$ be the characteristic function of C, that is, $\phi(t)=1$ when $t\in C$ and $\phi(t)=0$ otherwise; and let $\psi(t)$ be the characteristic function of B. Then $\psi(t-\lambda)$ is the characteristic function of $B(\lambda)$, and $\phi(t)\psi(t-\lambda)$ is the characteristic function of the intersection $CB(\lambda)$ of C and $B(\lambda)$. Hence on denoting the measure of $CB(\lambda)$ by $\mu(\lambda)$ we have

$$\mu(\lambda) = \int_{-\infty}^{\infty} \phi(t)\psi(t-\lambda)dt. \tag{12}$$

The function $\mu(\lambda)$ is continuous since

$$\begin{aligned} |\mu(\lambda+h)-\mu(\lambda)| &\leqq \int_{-\infty}^{\infty}\phi(t)\,|\psi(t-\lambda-h)-\psi(t-\lambda)|\,dt \\ &\leqq \int_{-\infty}^{\infty}|\psi(t-\lambda-h)-\psi(t-\lambda)|\,dt \\ &= \int_{-\infty}^{\infty}|\psi(t-h)-\psi(t)|\,dt \end{aligned}$$

and the last integral converges to 0 with h. Hence $\mu(\lambda)$ has a maximum over the interval $-1 \leqq \lambda \leqq 1$. Since $\mu(\lambda)=0$ when $|\lambda|>1$, the computation

$$\int_{-1}^{1} \mu(\lambda) d\lambda = \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} \phi(t)\psi(t-\lambda) dt$$
$$= \int_{-\infty}^{\infty} \phi(t) dt \int_{-\infty}^{\infty} \psi(t-\lambda) d\lambda = |C||B|$$

is easily justified. This equality and the inequality

(13) $$\mu(\lambda) \leqq \max_{-1 \leqq \lambda \leqq 1} \mu(\lambda), \qquad -1 \leqq \lambda \leqq 1,$$

imply that $\max |CB(\lambda)| = \max \mu(\lambda) \geqq \frac{1}{2}|C||B|$ and Lemma 1 is established.

The fact that use of inequalities such as (13) often leads to crude results may make one suspicious that Lemma 1 holds when the factor $\frac{1}{2}$ in (11) is replaced by a greater factor. To settle this question, let $0<\epsilon<\frac{1}{3}$, let $C=E_t\{\epsilon \leqq t \leqq 1-\epsilon\}$, and let $B=E_t\{0 \leqq t \leqq \epsilon\} + E_t\{1-\epsilon \leqq t \leqq 1\}$. Then $|C|=1-2\epsilon$, $|B|=2\epsilon$, and it is easy to verify that

(14) $$\max_{-1 \leqq \lambda \leqq 1} |CB(\lambda)| = \epsilon = [1/(2-4\epsilon)]|C||B| > 0.$$

This shows that $\frac{1}{2}$ is the greatest factor permissible in (11).

LEMMA 2. *If $A_1, A_2, \cdots$ is a sequence of measurable sets and a sequence $U_1, U_2, \cdots$ of unit intervals exists such that*

(15) $$\sum_{n=1}^{\infty} |U_n A_n| = \infty,$$

then there exists a sequence $\lambda_1, \lambda_2, \cdots$ such that each t in the interval $-\infty<t<\infty$, except those in some null set, lies in an infinite number of the sets $A_n(\lambda_n)$.

Let $B_n = U_n A_n$ so that each B_n lies in some unit interval and $\sum |B_n| = \infty$. Let n be fixed. Choose λ_n such that $B_n(\lambda_n) \subset U$, where U is as before the unit interval $0 \leqq t \leqq 1$, and let $C_n = U - B_n(\lambda_n)$. Since λ'_{n+1} exists such that $B_{n+1}(\lambda'_{n+1}) \subset U$, Lemma 1 guarantees existence of λ_{n+1} such that

(16) $$|C_n B_{n+1}(\lambda_{n+1})| \geqq \tfrac{1}{2}|C_n||B_{n+1}|.$$

Let $C_{n+1} = U - [B_n(\lambda_n) + UB_{n+1}(\lambda_{n+1})]$. Again from Lemma 1, λ_{n+2} exists such that (16) holds when n is replaced by $n+1$. In this manner, we obtain a sequence $\lambda_n, \lambda_{n+1}, \cdots$ of real numbers and a sequence

$$(17) \qquad C_{n+p} = U - [UB_n(\lambda_n) + UB_{n+1}(\lambda_{n+1}) + \cdots + UB_{n+p}(\lambda_{n+p})]$$

of sets such that, for each $p=0, 1, 2, \cdots$,

$$(18) \qquad \sum_{k=n}^{n+p} | C_k B_{k+1}(\lambda_{k+1}) | \geqq \frac{1}{2} \sum_{k=n}^{n+p} | C_k | | B_{k+1} |.$$

Since the sets $C_k B_{k+1}(\lambda_{k+1})$ $(k=n, n+1, \cdots, n+p)$ are subsets of U and no two have a point in common, the left member of (18) is less than or equal to unity for each $p=0, 1, 2, \cdots$. From this it follows that $|C_{n+p}| \to 0$ as $p \to \infty$; for $|C_{n+p}|$ is monotone decreasing as $p \to \infty$ and if $|C_{n+p}|$ is bounded from 0, then the fact that $\sum |B_n| = \infty$ would imply that the right member of (18) diverges to $+\infty$ as $p \to \infty$. The conclusion that $|C_{n+p}| \to 0$ as $p \to \infty$ implies by (17) that

$$(19) \qquad \lim_{p \to \infty} | UB_n(\lambda_n) + UB_{n+1}(\lambda_{n+1}) + \cdots + UB_{n+p}(\lambda_{n+p}) | = 1.$$

Hence there exists a sequence $0 = n_1 < n_2 < \cdots$ of indices such that the set

$$(20) \qquad D_k \equiv UB_{n_k+1}(\lambda_{n_k+1}) + \cdots + UB_{n_{k+1}}(\lambda_{n_{k+1}})$$

has measure $|D_k| > 1 - 2^{-k-1}$ for each $k = 1, 2, \cdots$. Put $P_k = D_k D_{k+1} \cdots$ and $P = P_1 + P_2 + \cdots$. The fact that $D_k \subset U$ and $|D_k| > 1 - 2^{-k-1}$ for each $k = 1, 2, \cdots$ implies that $P_k \subset U$ and $|P_k| \geqq 1 - 2^{-k}$, and consequently $P \subset U$ and $|P| = 1$. If $t \in P$, then $t \in P_k$ for some k so that $t \in D_k$ for all sufficiently great k and $t \in B_n(\lambda_n)$ for an infinite set of n, and hence also $t \in A_n(\lambda_n)$ for an infinite set of n.

If the sequence of sets A_n is arranged in a double sequence $A_{p,q}$ $(p = 0, \pm 1, \cdots; q = 1, 2, \cdots)$ in such a way that

$$(21) \qquad \sum_{q=1}^{\infty} | A_{p,q} | = \infty, \quad p = 0, \pm 1, \pm 2, \cdots,$$

it results from what we have already proved that for each fixed p there is a sequence $\lambda_{p,1}, \lambda_{p,2}, \cdots$ such that each point of a subset of $I_p \equiv E_t\{p \leqq t \leqq p+1\}$ of measure unity is contained in an infinite number of the sets $A_{p,1}(\lambda_{p,1}), A_{p,2}(\lambda_{p,2}), \cdots$. Then each point of $-\infty < t < \infty$ with the exception of a null set lies in an infinite number of sets of the double sequence $A_{p,q}(\lambda_{p,q})$ which can be arranged in the simple sequence $A_n(\lambda_n)$, and proof of Lemma 2 is complete.

The hypothesis of Lemma 2 is equivalent to the following: A_n is a sequence of measurable sets such that

$$\sum_{n=1}^{\infty} \underset{-\infty<h<\infty}{\text{l.u.b.}} \left| E_t\{h \leqq t \leqq h+1; t \,\varepsilon\, A_n\} \right| = \infty. \tag{22}$$

That the hypothesis (22) cannot be relaxed is a consequence of the following result which we give without proof. If $A_1, A_2, \cdots$ is a sequence of measurable sets, and a real sequence $\lambda_1, \lambda_2, \cdots$ and a set C of positive measure exist such that each point of C lies in an infinite number of the sets $A_n(\lambda_n)$, then (22) holds.

That the conclusion of Lemma 2 must provide for an exceptional null set becomes clear when one observes that if the sets A_n are each nondense then, however $\lambda_1, \lambda_2, \cdots$ are determined, the set $\sum A_n(\lambda_n)$ must be of the first category and hence there must be a set of the second category whose points are in *none* of the sets $A_n(\lambda_n)$.

4. **Proof of Theorem** 2. To prove Theorem 2, let $x_n(t)$ be a sequence of measurable functions for which (2) fails for some $\delta>0$. Then $\delta>0$ and a sequence $h_1, h_2, \cdots$ exist such that

$$\sum_{n=1}^{\infty} E_t\{h_n \leqq t \leqq h_n+1; \left| x_n(t) \right| \geqq \delta\} = \infty. \tag{23}$$

Let $A_n = E_t\{\left| x_n(t) \right| \geqq \delta\}$. Then by Lemma 2 there exist a sequence $\lambda_1, \lambda_2, \cdots$ and a set C whose complement is a null set such that each t in C lies in an infinite number of the sets $A_n(\lambda_n)$. Hence if $t \,\varepsilon\, C$, then $t-\lambda_n \,\varepsilon\, A_n$ for an infinite set of n so that $\left| x_n(t-\lambda_n) \right| \geqq \delta$ for an infinite set of n. This contradicts the hypothesis of Theorem 2 and completes the proof.

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